NOTES IN PREPARATION FOR A COURSE IN AXIOMATIC REASONING

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Contents

Our seminar course

I want to thank Giovanni Sommaruga for inviting me to include my notes in this volume. He suggested that it should comprise a “many facetted view on axiomatics, something like a tour d’horizon of the subject.” I had written these note in preparation for a seminar course that I co-taught with Amartya Sen and Eric Maskin. The title of the course was *Axiomatic Reasoning* and it followed two other course we three gave: *Reasoning via Models* and *Utilty*; and it preceded a course entitled *Subjectivity/ Objectivity*.

This, then, is not a standard article with *thesis, development, and conclusion*. There will be nothing that could be labelled ’original’ in it. It rather is a collection of notes and quotations meant as ‘starter’ to precede course readings, and to initiate reflections and discussions. The intent of our four courses was to create an opportunity for ourselves and our students to *live with* a concept for a significant length of time—without specifying a particular goal other than to become at home with, intimate with, the concept in broad terms in its various facts and its various moods. To become acquainted with a bit of the history of the concept, its reception, its development.

Such an experience can provide resonances which enrich thoughts that one may have, or can connect with ideas that one encounters, years later.[1] Our aim was to shape our seminar following the different experiences, background, viewpoints, and preferences of our students.

[1] This follows the format of some seminars I once taught with the late historian of Science John Murdoch. Murdoch would ask me at the beginning of the year:

What do you want to know?

and he would shape the readings and theme of the ensuing course often based on my answer. This “What do *you* want to *know*,” being abruptly personal, allowed the seminars to be somewhat oblique to the standard professional academic themes of discourse.

Part 1. Introduction to the theme of Axiomatic Reasoning

The etymological root of the word *axiom* is the Greek $\alpha\xi i\bar{o}\mu\alpha$ meaning 'what is fitting.' The concept *axiom* is often taken to mean 'self-evident assertion,' but we will take a much broader view, allowing it to encompass frame-creating assertions ranging from 'common notions' (Euclid's favorite) to 'rules,' 'postulates,' 'hypotheses,' and even definitions' if they play a suitable role in the ensuing discussion.

Axioms as a tool, a way of formulating a reasoned argument, a way of making explicit one's 'priors' or prior assumptions, a way of stipulating assertions very clearly so as to investigate their consequences, of organizing beliefs; a way of... in short, reasoning, has been with us for a long time, and in various guises.

Axiomatic frameworks offer striking transparency and help open to view the lurking assumptions and presumptions that might otherwise be unacknowledged. This mode of thought has been with us at least since Aristotle.

Axioms in formal (and even sometimes in somewhat informal) structures constitute an 'MO' of mathematics at least since Euclid, but surely earlier as well. "Surely," despite, curiously, the lack of any earlier record of it; and despite the fact that there is substantial record of much earlier mathematical thought. Egyptian mathematical papyri contain quite an array of problems and their solutions—e.g., the Moscow Mathematical Papyrus, dated approximately 1700 BC offers a correct discussion regarding the volume of a truncated square pyramid and a step-by-step computation of a particular example—but nowhere in these papyri is there as much as a hint of any mathematical protocol for demonstration, let alone any axiomatic foundational structure.

We will see how the very core of meaning and use of *axiom* in mathematics has undergone quite an evolution, through Euclid, his later commentators, Hilbert's revision of the notion of axiom, and the more contemporary set theorists.

Axioms are standard structures as they appear in models in the sciences, sometimes occurring as proclaimed 'laws': borrowing that word from its *legal* roots. Newton's Laws act as axioms for Classical Mechanics, the fundamental laws of thermodynamics for Thermodynamics.

Similarly for Economics: Axiomatic Utility Theory is very well named where the 'axioms' play more the role of *desiderata* which may or may not be realizable, especially in the face of the variously named 'paradoxes' and 'Impossibility Theorems' (as Professors Maskin and Sen will be discussing). Nevertheless this axiomatic format provides us with an enormously useful and powerful tool to understand forces at play in Economics.

Noam Chomsky's *Structural linguistics* set the stage for a very axiomatic approach to language acquisition and use (with interesting later critique) as does the vast tradition of rule-based grammars (as Professor Sen will be discussing later in this seminar).

Rules in games, and in the formal set-ups in mathematical game theory have their distinct qualities. Even more of 'distinct quality' is the subtle manner in which it is sometimes understood that rules are *not* expected to be strictly obeyed; e.g., as in the composition of a sonnet.

In the Bayesian mode of inductive reasoning, the 'priors' (as the Bayesians call them)—which are, in effect, input axioms—are constantly re-assessed in connection with the flow of further incoming data ("the data educates the priors" as they sometimes say). This is also quite a distinctive way of dealing with one's axioms!

We live these days at a time when computer programs, governed by *algorithms*—hence a specific form of axiomatic reasoning—make selections for us, recommendations, choices, and sometimes critical decisions. The question of when, and how, more flexible modes of human judgment should combine with, and possibly mitigate, axiom-driven decision processes is a daily concern, and something that we might address, at least a bit, in our seminar[2].

Most curiously, axiomatic structure has come up in various reflections regarding moral issues. (This, by the way, happens more than merely the *golden rule*: we might read a bit of Spinoza's *Ethics* which is set up in the formal mode of mathematical discourse, complete with Postulates, Definitions, and Theorems.)

Part 2. The evolution of definitions and axioms, from ancient Greek philosophy and mathematics to Hilbert.

Here is Socrates lecturing to Adeimantus in Plato's *Republic* VI.510c,d:

> *. . . the men who work in geometry, calculation, and the like treat as known the odd and the even, the figures,*

[2]Relevant to this discussion is Stephen Wolfram's *A New Kind of Science* (`https://www.wolframscience.com/`) in which it is proposed that one simply replace equations in scientific laws with algorithms.

> *three forms of angles, and other things akin to these in each kind of inquiry.*
>
> *These things they make* **hypotheses** *and don't think it worthwhile to give any further account of them to themselves or others as though they were clear to all. Beginning from them, they ... make the arguments for the sake of the square itself and the diagonal itself, not for the sake of the diagonal they draw, and likewise with the rest. These things themselves that they mold and draw—shadows and images in water—they now use as images, seeking to see those things themselves, that one can see in no other way than with thought.*

1. Venerable formats for reasoned argument and demonstration

Ancient organizational schemes of logic, such as the *Organon* of Aristotle, have been vastly influential and have been—even if largely implicit—the armature of the way in which we formulate assertions, ask questions, and reach conclusions in mathematics as in everything else. Aristotle begins his discussion in the *Prior Analytics* by setting for himself quite a task: to pin down *demonstration* "and for the sake of demonstrative science," to:

> ...define, what is a proposition, what a term, and what a syllogism, also what kind of syllogism is perfect, and what imperfect; lastly, what it is for a thing to be, or not to be...

Aristotle gets to this job right away, and offers this succinct definition of *proposition* neatly distinguishing between those propositions involving universal, no, or existential quantification:

> A proposition then is a sentence which affirms or denies something of something, and this is universal, or particular, or indefinite.

and, turning to *syllogism*, the main object of exploration of the *Organon*, he characterizes *syllogism* as:

> discourse in which, certain things being stated, something other than what is stated follows of necessity from their being so.

Since *definition*, defined by Aristotle as: *an account which signifies what it is* **to be** *for something*[3] plays such a vital role in mathematics, the notion deserves close attention. Mathematics seems to require as strict lack-of-ambiguity in its assertions as possible, and therefore maximal clarity in its definitions. But perhaps—since ambiguity is sometimes unavoidable—it is better to say that any ambiguity should be unambiguously labeled as such.

The nature, and role, of *definition* in mathematical usage has evolved in remarkable ways. We will be discussing this in more detail later, but consider the first two definitions in Book I of Euclid's *Elements:*[4]

(i) A point is that which has no part.
(ii) A line is breadthless length.

and their counterparts in Hilbert's rewriting of Euclid's *Elements*, which begins with:

> Let us consider three distinct systems of things. The things composing the first system, we will call **points** and designate them by the letters A, B, C,. . . ; those of the second, we will call **straight lines** and designate them by the letters a, b, c,. . . The points are called the **elements of linear geometry**; the points and straight lines, **the elements of plane geometry**. . .

Hilbert's undefined terms are: *point, line, plane, lie, between,* and *congruence.*

One might call Euclid's and Hilbert's formulations **primordial definitions** since they spring ab ovo–i.e., from nothing. Or at least from 'things' not in the formalized arena of mathematics, such as Hilbert's *"system of things"*. Euclid's definitions of *point* and *line* seem to be

[3] a puzzling definition: *logos ho to ti ên einai sêmainei*

[4] These 'Elements' have quite an impressive spread, starting with the proclamation that a point is characterized by the property of 'having no part,' and ending with its last three books, deep into the geometry of solids, their volumes, and the five Platonic solids. It is tempting to interpret this choice of ending for the *Elements* as something of a response to the curious interchange between Socrates and Glaucon in Plato's *Republic* (528a-d) where the issue was whether Solid Geometry should precede Astronomy, and whether the mathematicians had messed things up.

It also would be great to know exactly how—in contrast—the *Elements* of Hippocrates of Chios ended. (It was written over a century before Euclid's *Elements* but, unfortunately, has been lost.)

whittling these concepts into their pure form from some more materially graspable context (e.g., where lines have breadth)[5] while for Hilbert the essence of *point and line* is their relationship one to the other.

Once one allows the bedrock of—say—Set Theory, definitions are often 'delineations of structure,' cut out by means of quantifiers and predicates but making use of set theoretic, or at least priorly defined objects. E.g. *A circle is a set of points equi-distant from a single point in the Euclidean plane.* We will discuss this in a moment.

The essential roles that 'definition' play for us are: to delineate the objects of interest to be studied; to encapsulate; to abbreviate; and to focus.

2. The axiomatic 'method'

Axioms, as we've seen, have been around— at least—since ancient Greek mathematical activity, but only more recently have people viewed the act of 'listing axioms' as a *method,* rather than (somewhat more relaxedly) as a natural move to help systematize thought.

It may have been David Hilbert who actually introduced the phrase *axiomatic thinking* to signal the fundamental role that the structure of an axiomatic system plays in mathematics. Hilbert clearly views himself as molding a somewhat new architecture of mathematical organization in his 1918 article "Axiomatisches Denken." It begins with a political metaphor, that neighboring sciences being like neighboring nations need excellent internal order, but also good relations one with another, and:

> ... The essence of these relations and the ground of their fertility will be explained, I believe, if I sketch to you that general method of inquiry which appears to grow more and more significant in modern mathematics; the axiomatic method, I mean.

Hilbert's essay ends:

> In conclusion, I should like to summarize my general understanding of the axiomatic method in a few lines. I believe: **Everything that can be the object of scientific thinking in general, as soon as it is**

[5] I want to thank Eva Brann for pointing this out.

> **ripe to be formulated as a theory, runs into the axiomatic method and thereby indirectly to mathematics.** Forging ahead towards the ever deeper layers of axioms in the above sense we attain ever deepening insights into the essence of scientific thinking itself, and we become ever more clearly conscious of the unity of our knowledge. In the evidence of the axiomatic method, it seems, mathematics is summoned to play a leading role in science in general.

3. Formulating definitions and axioms: a beginning move.

In the "Definitions" of Euclid's *Elements* it is striking how these are both definitions (or at the very least descriptions) as well as axioms. Definitions, of course, suffer the risk of being 'ambiguous' and collections of axioms suffer the risk of being inconsistent or—in various ways–iinadequate. The essential roles that 'definition' play for us are: to delineate the objects of interest to be studied; to encapsulate; to abbreviate; and to focus.

As for the power of definition to provide 'focus,' consider the distinction between *definition* and *characterization*—as in the two equivalent definitions of prime number (given by (i) and (ii) below)—where one makes the choice to regard one of these as 'definition' and the other as 'characterization':

A prime number p is a (whole) number greater than one

(i) that is not expressible as the product of two smaller numbers.

or

(ii) having the property that if it divides a product of two smaller numbers, it divides one of them.

If you choose (ii) as the fundamental definition you are placing the notion of prime number in the broader context of 'prime'-ness as it applies to number systems more general than the ring of ordinary numbers—and more specifically in the context of *prime ideals* of a general ring. So choosing (ii) as definition casts (i) as a specific feature that characterizes prime numbers, thanks to the theorem that guarantees the equivalence of these to formulations. Going the other route—i.e., focusing on (i), the unfactorable quality of prime number, would then cast (ii) as a basic more general feature also characterizing prime-ness.

Even when axioms seem inadequate, or fail, the discussion can continue in an interesting way—very often dealing with more subjective issues. As Gödel's Incompleteness Theorem (see the discussion below) points to the striking limitations of Hilbert's grand notion of *formal system*, these limitations themselves have interesting implications. Similarly, vastly illuminating are the limitations implied by Arrow's Impossibility Theorem regarding social choice theory; or the 'named paradoxes" connected to axiomatic utility theory, such as:

- *Allais's Paradox* thatmanufactures a situation where the (ostensible) adding of 'equals' (i.e., adding further equal alternatives) to two choices that are open to us gets us to switch our preferred choice. Here the issue is that a *guaranteed* very large winning beats—in our assessment—a possible quite greater winning but with a 1% probability of total loss.

or

- *Ellsberg's Paradox* that points out the issue of *meta-risk assessment*—i.e., Ellsberg sets up a game where you must make a choice (A or B). And then compaes this game to a modified version with slightly different choices (A' or B') In the first game you actually know—i.e., can reasonably compute— the odds of winning depending on your choice; and you would choose A rather than B. Ellsberg then changes the game by modifying the two choices A or B *in an equal way* so that you are faced with choices A' or B' but the change is such that now you don't quite know the odds; and... curiously... now you would choose B'.

or

- *the St. Petersburg Paradox* already and the (much later) ideas of Kahnemann-Tversky—that emphasize the deeply subjective nature—and intertwining—of *utility* and *expectation*.

The evolution of the notion of axiom, the changes in formulation, and use, is striking. Compare the vastly different axiomatic formats of Euclidean geometry, as conceived by Euclid, David Hilbert, George Birkhoff, (and we'll also briefly mention below the viewpoint of the 'Erlangen Program' at least as it connects with the formulation of Euclidean Geometry).

4. A curious evolution

- **Euclid's Elements, Book I**

  (i) **Definitions**
  - (a) A point is that which has no part.
  - (b) A line is breadthless length.
  - (c) The extremities of a line are points.
  - (d) A straight line is a line which lies evenly with the points on itself.
  - (e) A surface is that which has length and breadth only.
  - (f) The extremities of a surface are lines.
  - (g) A plane surface is a surface which lies evenly with the straight lines on itself.
  - (h) (Def'n. 13) A boundary is that which is an extremity of anything.
  - (i) (Def'n. 14) A figure is that which is contained by any boundary or boundaries.
  - (j) (Def'n. 15) A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another;
  - (k) (Def'n. 16) And the point is called the centre of the circle.

  (ii) **Postulates**
  - (a) To draw a straight line from any point to any point.
  - (b) To produce a finite straight line continuously in a straight line.
  - (c) To describe a circle with any centre and distance.
  - (d) That all right angles are equal to one another.
  - (e) *('Fifth Postulate':)* That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

  (iii) **Common Notions**
  - (a) Things which are equal to the same thing are also equal to one another.
  - (b) If equals be added to equals, the wholes are equal.

(c) If equals be subtracted from equals, the remainders are equal.
(d) Things which coincide with one another are equal to one another.
(e) The whole is greater than the part.

– **Comparison with Archimedes**

Consider, for example, the treatise *On the Sphere and Cylinder* of Archimedes, where he lists five 'assumptions'—i.e., in effect: axioms—that he will depend upon:

(i) Of all the lines which have the same extremities the straight line is the least.
(ii) Of other lines in a plane and having the same extremities any two of them are unequal if they are both concave in the same direction and one is between the other and the straight line with the same extremities.
(iii) ... similar to (i) but defining plane among surfaces with the same ' extremities'.
(iv) ... similar to (ii) but distinguishing surfaces among surfaces with the same ' extremities'.
(v) *"Archimedes Principle":* Further of unequal lines, unequal planes, unequal solids, the greater exceeds the lesser by such a magnitude that when added to itself, can be made to exceed any assigned magnitude among those which are comparable...

– What are these assumptions of Archimedes? Descriptions, definitions, or axioms? Are they meant to be ad hoc—designed for the specific treatise in which they appear?

– **Issues of uniqueness** are perhaps implied both in Euclid, and Archimedes, but not specifically mentioned. E.g.:

* The following 'familiar' definitions, *not at all in the spirit of the definitions in Book I* appear in later interpretations of Euclid:

  A straight line is [implied: **uniquely**] *determined by two of its points.*

  Or:

A straight line segment is the [implied: **unique**] *curve of shortest distance between its end-points.*

   * Or uniqueness, for example, of that *central point* in Def'ns 15 and 16?

- **Substrate:** For me, the most striking fact about these definitions is that they don't rely on set theoretic vocabulary. We moderns immediately think 'sets,' 'subsets,' 'membership in sets,' etc. and tend to build our structures starting with sets as substrate.
- **Motion:** Euclid has no vocabulary *at all* for 'continuous motion,' 'transformation,' 'function' except as these issues are introduced in the *Postulates* and/or when one triangle is "applied" to another. *Discuss the 'Erlangen Program.'*

On the Postulates:

- The emphasis is on Construction rather than Existence.
- Regarding The Fifth Postulate: the minute one questions its independence, one is on the way to model-formation, but, of course, this is not at all in the spirit of Euclid.

On the Common Notions:

- These are closest to modern axiomatics, formulating rules regarding the terms *equal* and *greater than.* The intuitive notion—that, say, two angles are *equal* if there is a Euclidean transformation bringing one exactly onto the other—is utterly absent from Euclid's vocabulary.

• **Comparison with Nicholas of Oresmes' treatise:** *The Geometry of Qualities* **and with the analytic approach to geometry**

This deserves extensive discussion!

- **Hilbert's Euclidean Geometry**

Hilbert's axiom system is constructed with six **primitive notions:**

(i) three primitive terms:

- point;
- line;
- plane;

and

(ii) three primitive relations:

- *Betweenness,* a ternary relation linking points;
- *Lies on (Containment)*, three binary relations, one linking points and straight lines, one linking points and planes, and one linking straight lines and planes;
- *Congruence*, two binary relations, one linking line segments and one linking angles. Note that line segments, angles, and triangles may each be defined in terms of points and straight lines, using the relations of betweenness and containment. All points, straight lines, and planes in the following axioms are distinct unless otherwise stated.

And there are these structures, axioms, and lists of 'defined terms":

(iii) **Incidence:** For every two points A and B there exists a line a that contains them both...

(iv) **Order:** If a point B lies between points A and C, B is also between C and A, and there exists a line containing the distinct points A,B,C ... If A and C are two points of a line, then there exists at least one point B lying between A and C. Of any three points situated on a line, there is no more than one which lies between the other two.

(v) **Pasch's Axiom:** Let A, B, C be three points not lying in the same line and let L be a line lying in the plane ABC and not passing through any of the points A, B, C. Then, if the line L passes through a point of the segment AB, it will also pass through either a point of the segment BC or a point of the segment AC.

(vi) **Axiom of Parallels:** Let m be any line and A be a point not on it. Then there is a unique line in the plane, determined by m and A, that passes through A and does not intersect m.

(vii) **Congruence:** If A, B are two points on a line L, and if A' is a point upon the same or another line L', then, upon a given side of A' on the straight line L', we can always find a point B' so that the segment AB is congruent to the segment $A'B \dots$

(viii) **Continuity:**
- *Axiom of Archimedes:* If AB and CD are any segments then there exists a number n such that n segments CD constructed contiguously from A, along the ray from A through B, will pass beyond the point B.

- *Axiom of line completeness:* An extension of a set of points on a line with its order and congruence relations that would preserve the relations existing among the original elements as well as the fundamental properties of line order and congruence that follows from [the axioms discussed] is impossible.
- *Defined Terms: segment, ray, interior, triangle, 'lie on the same side', . . .*

- **George Birkhoff's Axioms for Euclidean Geometry**

Undefined Elements and Relations:

- points A, $B, \ldots$
- sets of points called lines ℓ, $m \ldots$
- distance between any two points A and B: a nonnegative real number $d(A, B)$ such that $d(A, B) = d(B, A)$.
- angle formed by three ordered points $A, O, B (A \neq O, B \neq O)$: $\angle AOB$ **a real number (mod 2π)**. The point O is called the **vertex** of the angle.

- **Postulate I.** (Postulate of Line Measure) **The points** $A, B, \ldots$ **of any line ℓ can be placed into one-to-one correspondence with the real numbers**, so that for x a non-negative real number, $|xA - xB| = xd(A, B)$ for all points A, B.

- **Postulate II.** (Point-Line Postulate) One and only one line ℓ contains two given points $P, Q \ \ (P \neq Q)$.

- **Postulate III.** (Postulate of Angle Measure) **The half-lines ℓ, $m \ldots$ through any point O can be put into one-to-one correspondence with the real numbers** a**(mod 2π),** so that, if $A \neq O$ and $B \neq O$ are points of ℓ and m, respectively, the difference $am - al \pmod{2\pi}$ is $\angle AOB$. Furthermore if the point B on m varies continuously in a line ℓ not containing the vertex O, the number am varies continuously also.

- **Postulate IV.** (Similarity Postulate) If in two triangles
$$\Delta ABC, \Delta A'B'C'$$
and for some constant $k > 0$,
$$d(A', B') = kd(A, B), \quad d(A', C') = kd(A, C),$$
and
$$\angle B'A'C' = \pm\angle BAC,$$
then also
$$d(B', C') = kd(B, C), \angle A'B'C' = \pm\angle ABC,$$
and
$$\angle A'C'B' = \pm\angle ACB.$$

Defined Terms:

A point B is between A and C $(A \neq C)$, if $d(A, B) + d(B, C) = d(A, C)$. The half-line ℓ' with endpoint O is defined by two points O, A in line ℓ $(A \neq O)$ as the set of all points A' of ℓ such that O is not between A and A'. The points A and C, together with all point B between A and C, for segment AC. If A, B, C are three distinct points, the segments AB, BC, CA are said to form a triangle ΔABC with sides AB, BC, CA and vertices A, B, C.

Topics that we might discuss regarding Hilbert:

- Hilbert's Axioms offer an articulation very different from Euclid's: the triple *definitions/postulates/common notions* being replaced by *primitive terms/primitive relations/structures and axioms*.
- The common notions (i.e., logical pre-structures like 'equality') are implicitly assumed rather than formulated.
- Modern quantification is explicit. E.g., the 'incidence Axiom' calls up universal and existential quantification: $\forall$ points A, B, $\exists$ a line through A and B.
- **Geometry as a structure,** following the Erlangen Program.
- **Set Theory:** Most importantly, Hilbert expresses his axioms in Set theoretic vocabulary.

  But if one uses Set Theory as a 'substrate' on which to build the structures of mathematics, as in the classical *Grundlagen der Mathematik* of Bernays and Hilbert, one must tangle with all the definitional questions that are faced by Set Theory (starting with: *what is a set?* and continuing with the discussion generated by the work of Frege, Russell, etc.) [6].

- **Infinity.** And then compare all this with the discussion about the existence of infinite sets in Bernays-Hilbert's *Grundlagen der Mathematik, Vol. I*:

  > . . . reference to non-mathematical objects can not settle the question whether an infinite manifold exists;

[6] For example, go back to Dedekind's marvelous idea of capturing the notion of *infinite* by discussing self-maps (this notion popularized by people checking into Hilbert's hotel). You might formulate Dedekind's idea this way: a set S is **infinite** if it admits an injective but non-surjective self-map. . . and then confuse yourself by trying to figure out how this compares with the property that S admits a surjective but non-injective self-map.

the question must be solved within mathematics itself. But how should one make a start with such a solution? At first glance it seems that something impossible is being demanded here: to present infinitely many individuals is impossible in principle; therefore **an infinite domain of individuals as such can only be indicated through its structure, i.e., through relations holding among its elements.** In other words: a proof must be given that for this domain certain formal relations can be satisfied. The existence of an infinite domain of individuals can not be represented in any other way than through the satisfiability of certain logical formulas...

From Hilbert's *On The Infinite:*

...We encounter a completely different and quite unique conception of the notion of infinity in the important and fruitful method of ideal elements. The method of ideal elements is used even in elementary plane geometry. *The points and straight lines of the plane originally are real, actually existent objects.* One of the axioms that hold for them is the axiom of connection: one and only one straight line passes through two points. It follows from this axiom that two straight lines intersect at most at one point. There is no theorem that two straight lines always intersect at some point, however, for the two straight lines might well be parallel. Still we know that by introducing ideal elements, viz., infinitely long lines and points at infinity, we can make the theorem that two straight lines always intersect at one and only one point come out universally true. These ideal "infinite" elements have the advantage of making the system of connection laws as simple and perspicuous as possible. Moreover, because of the symmetry between a point and a straight line, there results the very fruitful principle of duality for geometry....

Topics that we might discuss regarding Birkhoff:

- The big distinction between the three 'Euclidean axiom-formulations' (Euclid's, Hilbert's, and Birkhoff's) is—I believe–in the implicitly assumed substrates that ground each of the axiom systems:
  - Euclid *assumes* that we are—at least vaguely—familiar with the basic nature of 'Euclidean Space' and his mission is to describe it more precisely and give terminology so that we may offer reasoned arguments about its features and make constructions in it. *Discuss: proportions versus numbers.*
  - Hilbert—in effect—generates his 'Euclidean space' by relational axioms, depending on the substrate (undiscussed explicitly) of set theory.
  - Birkhoff brings in (in a way fundamental to his approach) metric considerations; hence his 'substrate' includes quite explicitly the system of real numbers.

5. From Synthetic to Analytic

All three axiom systems fall under the general rubric of "synthetic geometries," i.e., set-ups that formulate conditions regarding essentially geometric features. Note that there is *no* mention of the *real numbers* or any other number system in Euclid's or Hilbert's Axioms; this is not true of Birkhoff's axioms. All three systems are quite different from *'analytic geometry'* which would set things up—from the start—by working in the substrate of $\mathbb{R}^2$ or $\mathbb{R}^3$ and providing i geometric definitions in purely algebraic language. Birkhoff's axioms do move closer to that, but are still (interestingly) synthetic.

Here's Felix Klein's definition of the distinction between analytic and synthetic geometry:

> Synthetic geometry is that which studies figures as such, without recourse to formulas, whereas analytic geometry consistently makes use of such formulas as can be written down after the adoption of an appropriate system of coordinates.

and here are his comments in an essay he wrote:

> **On the Antithesis between the Synthetic and the Analytic Method in Modern Geometry:**
>
> The distinction between modern synthesis and modern analytic geometry must no longer be regarded as essential, inasmuch as both subject-matter and methods of reasoning have gradually taken a similar form in both. We choose therefore in the text as common designation of them both the term projective geometry. Although the synthetic method has more to do with space-perception and thereby imparts a rare charm to its first simple developments, the realm of space-perception is nevertheless not closed to the analytic method, and the formulae of analytic geometry can be looked upon as a precise and perspicuous statement of geometrical relations.

Also, when I was first learning geometry, Herbert Buseman was the main proponent of keeping as much "synthetic geometry" as possible, but even he realized that he was being out-dated as a purist. He wrote:

> Although reluctantly, geometers must admit that the beauty of synthetic geometry has lost its appeal for the new generation. The reasons are clear: not so long ago synthetic geometry was the only field in which the reasoning proceeded strictly from axioms, whereas this appeal so fundamental to many mathematically interested people is now made by many other fields.

There's a good Wikipedia page about this: `https://en.wikipedia.org/wiki/Synthetic_geometry`

Given all these sentiments, I take Birkhoff's axioms as being something of a compromise: it is largely synthetic, but with an analytic flavor.

In contrast to all three axiom systems, one has *The Erlangen Program.*

Discuss...

6. From Axioms to Models: example of hyperbolic geometry

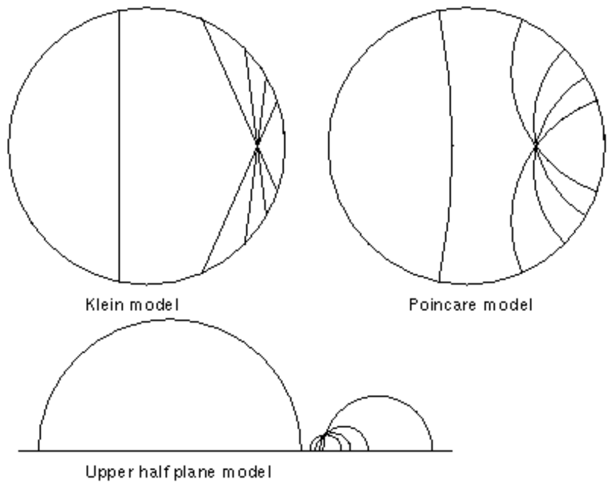
Often—when we frame an axiomatic system—- we have a specific structure in mind (Euclid surely did!) and our axiomatic system is a way of allowing us to understand, to study, the structure. Once the axiomatic system is formulated, though, we can reverse the process and ask for concrete structures that *model*[7] (or perhaps are modeled by) our axiomatic system. There are celebrated stories about this issue (*non*-Euclidean geometry) where various slight change of one postulate ("Euclid's Fifth Postulate') provides axiomatics for various different geometries—Hyperbolic Geometry being one of them.

Essential roles played by models (vis-á-vis axiomatic systems) is that

- the axiomatic systems may elucidate the models;
- the models will establish the *consistency* of the axioms (i.e., proving that they're not self-contradictory);
- the models will offer the intuition needed to think constructively about the axioms.

All this is nicely illustrated by the example of the multiplicity of different models for the same system of axioms of Hyperbolic Geometry. With Hyperbolic Geometry we have an assortment of different *models* any one of which conforms to the axiomatic system of Hyperbolic Geometry and we display them below. And this is a possible topic for our discussion.

[7] *Model Theory,* a subject we will touch on later focuses even more explicitly on the relation between setting-up a language-and-structure and models for such.



- *Klein model (used by many)*

  Geodesics are straight lines in the (open) unit disc.

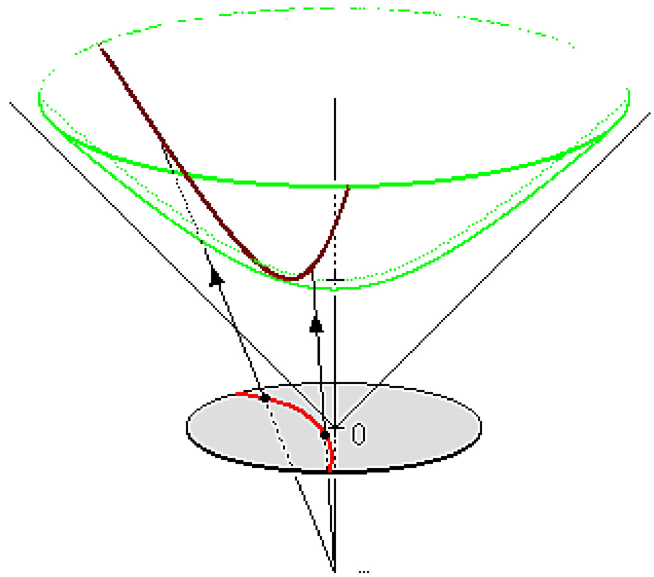
- *Poincaré disc model* (loved by geometers)

  Geodesics are arcs of circles perpendicular to the boundary.

- *Poincaré half-plane model* (loved by number theorists)

  Geodesics are vertical lines from the real axis to infinity; or semicircles perpendicular to the real axis.

- *Lorentz model* or *hyperboloid model* (loved by physicists)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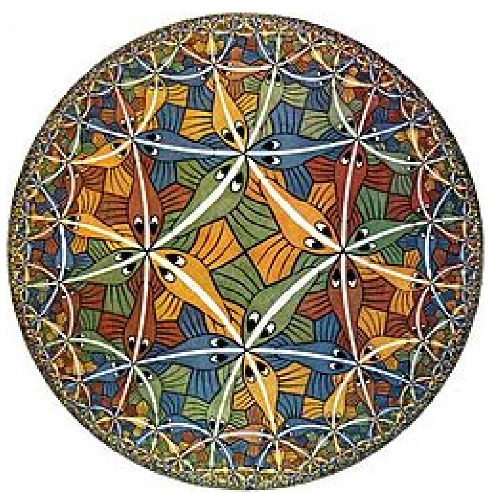
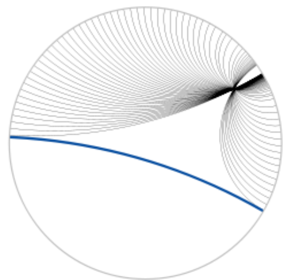
This coral, I'm told, is *very* approximately a hyperbolic planar surface *embedded* in Euclidean space:

M. C. Escher's *Circle Limit III* in which "strings of fish shoot up like rockets from infinitely far away" and then "fall back again whence they came." (This approximates a tiling by 'hyperbolical equilateral triangles and squares' represented in the Poincaré disc model.)



Here are lines through a given point and parallel to a given line, illustrated in the Poincaré disk model:



A good discussion of this is in `en.wikipedia.org/wiki/Hyperbolic_geometry`

Part 3. **'Axiomatic formats' in philosophy, Formal logic, and issues regarding foundation(s) of mathematics and... axioms in theology**

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7. Axioms, again

It may pay, at this point, to summarize the various uses we have seen of *the axiomatic format*.

(i) *To provide an explicit organized framework for thought:* axioms may be framed to set out the starting assumptions in a line of reasoning. Or, more formally, to formulate a specific algorithmic procedure.

(ii) *To describe in as explicit terms as possible certain specific human interactions:* either fully descriptive (e.g., this *is*—or *may be* a rough model of what people do), or more normative, *desiderata* perhaps. This is as in axiomatic utility theory.

(iii) *To stipulate a 'mathematical structure':* as in axioms for Euclidean or Hyperbolic Geometry. This can be descriptive—i.e., a characterization of a geometry (in the style of Euclid) or more—one might say—ontological (in the style of Hilbert) where the axioms are meant to be an abstract structure that has, as one of its features, Euclidean Geometry as (a) model. It is a structure that *defines* Euclidean Geometry.

(iv) *To 'delineate" a mathematical structure from a previously constructed axiomatic system.* A good example being:

Analytic Geometry:

- A **point** is (given by) a couple (x, y) where x and y are real numbers; i.e., the **plane** is (given by) the 2-dimensional vector space $\mathbf{R}^2$ over the field $\mathbf{R}$ of real numbers.
- A **line** is the locus of a "linear equation"

$$y = ax + b$$

  where $a, b \in \mathbf{R}$.
- etc.. . .

In a sense Birkhoff's axiom system is something of a hybrid synthetic/analytic axiomatic set-up.

8. Axioms... and 'psychology'

In axiomatic formulations of models for utility—e.g. axiomatic utilitarianism; or in attempts to model the manner in which we—individually, or collectively—make choices, it would seem that the *axioms* play an essential role as the *starting point* of a discussion—the axioms reflecting the seemingly-rational way that people do (or should) behave in making their individual or collective choices.

However,—crudely speaking—*psychology often intervenes,* making these axioms not entirely reliable predictors of behavior, and softening their effect as normative signposts. This is illuminated by the named *paradoxes* (Ellsberg, Allais, St. Petersburg) of Social choice theory, and by our reading of Kahneman & Tversky's *Rational choice and the framing of decisions* where among many other things, they show that

- two formulations of the same problem may elicit different preferences, in violation of the axiom of *invariance.*
- The *dominance rule* is obeyed when its application is transparent, but *dominance* can easily be "masked by a frame in which the inferior option yields a more favorable out- come in an identified state of the world."

  > Whether the relation of dominance is detected depends on framing as well as on the sophistication and experience of the decision maker.

In Kahneman & Tversky's language: different *framings* call into play different personal 'takes' on the axioms. A transformation, then, from *rules of rational choice* to the more malleable *rules for subjective reasonable choice.* This was already implicit in Daniel Bernoulli's notion of the concavity of the (function describing) the value of goods, or money. Of course, Kahneman & Tversky go much further in considering 'subjective takes on things' (and in many directions: even in their **weight functions** which might be more descriptively labelled as *subjective evaluation of probabilities*).

9. A crash course in Formal Logic

A. Formal Systems. A *Formal System* consists of:

- A finite set of *symbols,* that can be used for constructing formulas (i.e. finite strings of symbols).
- A *grammar*, which tells how well-formed formulas (abbreviated wff) are constructed out of the symbols in the alphabet. It is usually required that there be a decision procedure for deciding whether a formula is well formed or not.
- A set of *axioms* or axiom schemata: each axiom must be a wff.
- A set of *inference rules.*

B. Propositional Calculus (PC). The apparatus:

- symbols for 'variables:' $p, q, \ldots$
- 'operators:'

$$\neg,$$

$$\cdot \ (\text{or } \vee),$$

$$\wedge,$$

$$\supset \ (\text{or } \to \text{ or } \implies),$$

$$=$$

- 'brackets:' ( )

The 'meaning' of the operators are (in order):
$\neg$ means *not,*
$\cdot$ means *and,*
$\wedge$ means *or,*

$\supset$ means *implies;* and
$=$ means *equivalent in truth-value.*

The point about *truth-value* is that there is indeed a sort of calculus where—if you interpret each variable-symbol (e.g. p) as standing for the statement: p *is true* there are natural rules that guarantee that certain statements formulated with the above apparatus are true (or false). For example: if p is true, then—in this calculus— $\neg p$ is false, and vice versa.

The brackets are pretty important. For example, there's no associative law: $p \cdot (q \wedge r)$ is not the same as $(p \cdot q) \wedge r$, the former meaning

$$p \text{ and either } q \text{ or } r$$

while the latter means: p and q, or r.

Some inference rules:

- **modus ponens:** $\frac{a,\ a \Longrightarrow b}{b}$ (and $\frac{b}{a \Longrightarrow b}$)
- **and-addition; and-elimination:** $\frac{a,\ b}{a \vee b}$; $\frac{a \vee b}{a,\ b}$
- **or-introduction:** $\frac{a}{a \wedge b}$
- **Double-negative-reduction:** $\frac{\neg\neg a}{a}$ (and $\frac{a}{\neg\neg a}$)
- **Various 'resolutions:'** $\frac{a \wedge b, \neg a}{b}$, etc.

Discuss: wff, consistency, completeness, decidability

Example: Two symbols, p, q both taken as *axioms*. Then (e.g.,) any wff that doesn't have a $\neg$ sign in it is *true.*

$$p, \qquad q, \qquad p \wedge q, \qquad p \vee q, \qquad (p \implies q), \qquad p \wedge (p \vee q), \qquad \ldots$$

But there isn't much one can do with Propositional Calculus. One needs a richer structure:

C. First-order logic. This provides us with apparatus with which we can actually work. The new ingredients:

First there should be some domain of discourse—call it Ω—which I won't describe too explicitly, but then:

- *Terms:* By an **elementary term** one means a member of that stipulated domain of discourse. The set of **terms** is then inductively defined by the following rules:
  (i) *Elementary terms.*
  (ii) *Variables.* Any variable is a term.
  (iii) *Functions.* Any expression $f(t_1, t_2, \dots, t_n)$ of n arguments (where each argument is a term) and f is a *function symbol* of valence n) is a term.
- *Predicates.* These are, in effect, statements with a free variable (or a finite number of independent free variables) contained in them. A predicate should be viewed as a specific quality that a term might possess.

  For example, suppose the domain of discourse is the set of real numbers. Let x denote a 'free variable' and the predicate $P(x)$ is: "is positive." That is, it is short for the sentence 'x *is positive".* where the "x" here is taken to be a variable that might range though members of 'the domain of discourse.' So, fixing on some such member (e.g., π) you get the formula $P(\pi)$: $\pi > 0$. Another example: consider the predicate $P(x, y)$ that stands for the sentence "x is greater than y" where the domain of discourse, again, is the set of real numbers. Often predicates are simply associated with the sets of terms that are described by the sentence that 'interprets' the predicate.

- *Quantifiers* $\forall$, $\exists$; e.g., these quantifiers 'quantify' the predicates; they will be forming sentences such as:

$$\forall x \in \mathbf{R} \mid x^2 \geq 0.$$

- *Formulas*

The set of formulas (also called well-formed formulas or WFFs) is inductively defined by the following rules:

(i) If $P(x_1, x_2, \dots, x_n)$ is an" n-ary predicate symbol and $(a_1, a_2, \dots, a_n)$ are terms then $P(a_1, a_2, \dots, a_n)$ is a formula.
(ii) If a, b are terms, then $a = b$ is a formula.
(iii) If Φ is a formula, then $\neg\Phi$ is a formula.
(iv) If Φ and Ψ are formulas, then $(\Phi \wedge \Psi)$, $(\Phi \vee \Psi)$ and $(\Phi \rightarrow \Psi)$ are formulas.
(v) If φ is a formula and x is a variable, then $\forall x \varphi$—i.e., (for all x, φ holds)—is a formula.
(vi) If φ is a formula and x is a variable, then $\exists x \varphi$—i.e., (there exists a term a such that $\varphi(a)$ holds)—is a formula.

Only expressions which can be obtained by finitely many applications of rules (i)-(vi) are formulas. The formulas obtained from the first two rules are said to be atomic formulas.

One can also make sense of 'higher-order' theories, where—say for 'second-order logic' predicates may be interpreted as sets of sets, and quantification can be over sets (as well as members of sets).

Exercise: What 'logical system' do you need to define the *natural numbers*

$$\mathbf{N} := \{1, 2, 3, \ldots, \}?$$

E.g., We might do it this way:

Definition 1. $\mathbf{N}$ *is a set*

(i) *containing an element* $1 \in \mathbf{N}$ *and*
(ii) *admitting a mapping (call it the successor mapping)*

$$s : \mathbf{N} \to \mathbf{N}$$

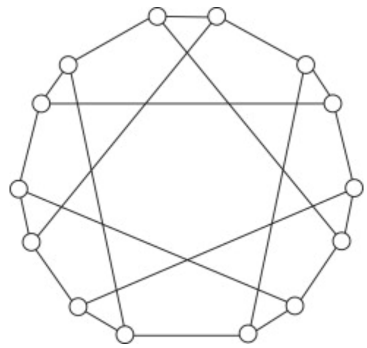
that is
(iii) *injective (i.e.,* $\forall x_1, x_2 \in \mathbf{N}$ if $x_1 \neq x_2$ then $s(x_1) \neq s(x_2)$*) and*
(iv) *the element* 1 *is not in the image of* s *and*
(v) $\mathbf{N}$ *is minimal with respect to the above properties in the sense that any subset* $\mathbf{N}' \subset \mathbf{N}$ *containing* 1 *and stable under the successor mapping* s *is equal to* $\mathbf{N}$.

So, describe the formal system necessary to encompass this axiomatic format.

10. Model Theory

Model theory begins by offering a format for doing mathematics within an explicitly shaped 'Language' (in the style of 'universal algebra')—where the 'models' will be *sets with extra structure*—and where its *sentences* interpreted in any 'model' have truth-values that conform to the rules of first-order logic.

The 'opening move' of Model Theory is a powerful and revealing disarticulation of semantics from substance. Here's what I mean: if you are not model-theoretic and want to formulate, say, *graph theory*, you might—for example—just define a **graph** to be given by a set V of vertices and a set E of edges, each edge attaching two distinct vertices and you might also insist that no two vertices are attached by more than one edge. Or you might give a more topological account of this



structure.

In any event, your formulation begins with a set and then some structure is imposed on it.

Model Theory, reverses this. It begins by offering an explicitly shaped language in which first-order logic is incorporated. In the case of our example of graph theory, the language would have a symbol $\mathcal{E}$ labeled as a *binary relation* (symmetric, but not reflexive) in connection with which we label as *true* sentences:

$$\forall x, y(x\mathcal{E}y \leftrightarrow y\mathcal{E}x)$$

and

$$x\mathcal{E}y \implies x \neq y.$$

An 'interpretation' of this language—or synonymously, a 'model' for this would be a 'representation' of this language in (some version of) Set Theory. That is, it would give us a set V endowed with a binary relation E for which the labeled-as-true sentences are... in fact true; i.e., such a model is simply a graph, where the set of vertices is the set V and the set of edges is given by the binary relation E.

11. Completeness, Consistency

A Formal System (consisting in, say, a first order theory with a finite collection of axioms) is **consistent** if it is not the case that there exists a wff P such that P and $\neg P$ can both be proved from those axioms. It is **complete** if for every wff P either P or $\neg P$ can be proved.

...Discussion...

Take a look—just a look—at Martin Davis, *Gödel's incompleteness Theorem*, Notices of the AMS **53** (2006) 414-418 `http://www.ams.org/notices/200604/fea-davis.pdf`

12. Axiomatic language in Ethics

There is much to discuss here: It is (essentially) the Euclidean format for the organization of rational argument—including first principles of a very general nature (often labelled *Definitions* and *Axioms*) formally set down and referred to explicitly in the justification of each step of arguments that have played important roles in moral philosophy. For example:

A. From: Baruch de Spinoza, ***Ethics*****.**
The organization of his *Ethics* has a Euclidean framework where there are very explicitly displayed *Definitions*, *Axioms*, and *Propositions* and where the propositions have arguments labeled *Proof* in which every line of argument refers only to prior *Definitions*, *Axioms*, or *Propositions*.[8].

The long quotation below is from PART II of Spinoza's Ethics:

On the Nature and Origin of the Mind

First, I would like some discussion about this, since I don't have any idea how Spinoza expects us to deal with this 'borrowed' Euclidean format:

> **DEFINITION I.** By body I mean a mode which expresses in a certain determinate manner the essence of God, in so far as he is considered as an extended thing. (See Pt. i., Prop. xxv., Coroll.)
>
> **DEFINITION II.** I consider as belonging to the essence of a thing that, which being given, the thing is necessarily given also, and, which being removed, the thing is necessarily removed also; in other words, that without which the thing, and which itself without the thing, can neither be nor be conceived.
>
> **DEFINITION III.** By idea, I mean the mental conception which is formed by the mind as a thinking thing.
>
> **Explanation:** I say conception rather than perception, because the word perception seems to imply that

[8] See `https://www.gutenberg.org/files/3800/3800-h/3800-h.htm`

the mind is passive in respect to the object; whereas conception seems to express an activity of the mind.

DEFINITION VI. Reality and perfection I use as synonymous terms.

...

AXIOMS

...

AXIOM II. Man thinks.

AXIOM III. Modes of thinking, such as love, desire, or any other of the passions, do not take place, unless there be in the same individual an idea of the thing loved, desired, etc. But the idea can exist without the presence of any other mode of thinking.

...

PROPOSITIONS

PROP. I. Thought is an attribute of God, or God is a thinking thing.

Proof: Particular thoughts, or this and that thought, are modes which, in a certain conditioned manner, express the nature of God (Pt. i., Prop. xxv., Coroll.). God therefore possesses the attribute (Pt. i., Def. v.) of which the concept is involved in all particular thoughts, which latter are conceived thereby. Thought, therefore, is one of the infinite attributes of God, which express God's eternal and infinite essence (Pt. i., Def. vi.). In other words, God is a thinking thing. **Q.E.D.**

Spinoza ends his treatise *Ethics* in a curious manner, offering a a quite different take than St. Anselm's proof of the existence of God— or indeed on the ontological proofs offered by Gödel:

> PROP. XIII. A mental image is more often vivid, in proportion as it is associated with a greater number of other images.
>
> Proof. In proportion as an image is associated with a greater number of other images, so (II. xviii.) are there more causes whereby it can be aroused. Q.E.D.
>
> PROP. XIV. The mind can bring it about, that all bodily modifications or images of things may be referred to the idea of God.
>
> Proof. There is no modification of the body, whereof the mind may not form some clear and distinct conception (V. iv.); wherefore it can bring it about, that they should all be referred to the idea of God (I. xv.). Q.E.D.
>
> PROP. XV. He who clearly and distinctly understands himself and his emotions loves God, and so much the more in proportion as he more understands himself and his emotions.
>
> Proof. He who clearly and distinctly understands himself and his emotions feels pleasure (III. liii.), and this pleasure is (by the last Prop.) accompanied by the idea of God; therefore (Def. of the Emotions, vi.) such an one loves God, and (for the same reason) so much the more in proportion as he more understands himself and his emotions. Q.E.D.
>
> PROP. XVI. This love towards God must hold the chief place in the mind.

B. The Categorical Imperative. Perhaps the most quoted example of axiomatics in ethics is Immanuel Kant's 'golden rule': the categorical imperative as formulated in his *Critique of Practical Reason*.[9].

> Act in such a way that the maxim of your will can at the same time always hold as a principle of a universal legislation.

[9] See page 38 (**Fundamental Principle of Pure Practical Reason**) in `http://www.kantwesley.com/Kant/CritiqueOfPracticalReason.pdf`

This succinct proposition expands impressively in its applications, and is meant to fit into the context of various *Theorems* of morality stated by Kant. Kant insists on the imperative here, noting that "Pure geometry has postulates as practical propositions which, however, contain nothing further than the presupposition that one is *able* to do something if it were required." Geometry's propositions are, Kant says, "therefore, practical rules under a problematical condition of the will." (What Kant is signaling here is that in Geometry one can construct a circle *or not:* there's no obligation to perform any act—but his theorems of morality oblige 'the will' to act in the above way.)

Earlier in his *Introduction* Kant writes:

> For in the present work we will begin with principles and go to concepts, and only then from these, where possible, continue on to the senses. With speculative reason, in contrast, we began with the senses and had to end with the principles.

which I take to mean that even though he works in a formal setting (with statements labeled as "Theorems," etc.) he is inverting the usual order of appearance of elements in a formal system—starting with conclusions ("principles") and having them reveal basic concepts (an *analysis* of propositional truths rather than a *synthesis*).

C. Ontological Arguments. But the most curious engagement of axiomatics in ethics are the various "Ontological arguments" related to the existence (or at least to the definition) of God. These arguments can be essentially pro- (i.e., claiming that God's existence is proved) as in St. Anselm, or Spinoza; or essentially critical as in Aquinas or Kant. An enlightening account of these arguments, and their history, can be found in the Stanford Encyclopedia of Philosophy's entry `https://plato.stanford.edu/entries/ontological-arguments/`. (Read especially sections 1-3.)

A curious common thread in many of the ontological arguments is to allow 'existence' to be a possible predicate (or not!) of the various things-of-thought. St. Anselm, for example, puts a value judgment on this predicate :*it's more perfect to exist than not!*[10]. Compare this with Spinoza's Definition IV quoted above.

[10]*Old joke:*
A: I wish I never was born!
B: Oh, only one in a million is that lucky.

So if you conjure the most perfect thing-of-thought that can be conceived, well: if it doesn't exist, there's your contradiction. For now imagine whatever it is that you conjured up, but as *existing*, and you've just conceived of a yet more perfect thing-of-thought—voilá.

Often in these ontological arguments one sees the *unqualified* use of the quantifier $\exists$ to establish *existence* (of something) as being a predicate (of that something). That is: one asserts existence of an entity, without specifying in what realm that entity is (so-to-speak) 'taken from.'

In symbols: as long as you have a set in mind as your domain of discourse—call it Ω—it makes sense to consider formulas such as:

$$\exists x \in \Omega \;\text{ such that } \ldots,$$

but you're asking for trouble if you have no specific set such as Ω in mind and just want to deal with the formula:

$$\exists x \;\text{ such that } \ldots.$$

(This puts such arguments in the same framework as *unqualified* use of the quantifier $\forall$, as is behind Russell's paradox and the various uses of 'unqualified universal quantification, related to the classical *crisis in the foundations of mathematics*.)

Baruch de Spinoza, however, in his *Ethics* has—as far as I can make out—a very different take. Spinoza gives three different 'proofs'. These might be characterized as follows.

(i) Two versions of the 'ontological argument':

- God's essence (simply) entails existence.
- > The potentiality of non-existence is a negation of power, and contrariwise the potentiality of existence is a power, as is obvious.

(ii) A version of the principle of insufficient reason:

> If, then, no cause or reason can be given, which prevents the existence of God, or which destroys his existence, we must certainly conclude that he necessarily does exist. If such a reason or cause should be given, it must either be drawn from the very nature of God, or be external to him—that is, drawn from another substance of another nature. For if it were of the same nature, God, by that very fact, would be admitted to exist. But substance of another nature

> could have nothing in common with God (by Prop. ii.), and therefore would be unable either to cause or to destroy his existence.

13. Axiom-Definitions in Classical Physics

We have already encountered instances of 'organizing statements' setting up an axiomatic framework, where it isn't quite clear to what extent a statement is a *definition* of something or an *axiom* about the behavior of something[11]. A simple example of this is the labelled Definition 16 in Euclid's *Elements*: *And the point is called the centre of the circle.* That this point is 'unique' has 'gone without saying,' i.e., the definition carries along with it the uniqueness assertion, as axiom.

Newtons three Laws form–or at least suggest—a formal axiom system. Specifically interesting for our seminar is the structure, the nature, and various historical revisions imposed on these axioms. Moreover, at times these laws move from their status as axiom (the status they enjoy in straight Newtonian mechanics) to *part-definition* or to something less easily describable. Here are Newton's Laws, not in their original language, but in their original strength and intent as they were given in Newton's 1687 treatise: *Mathematical Principles of Natural Philosophy*:

- **First law:** In an inertial frame of reference, an object either remains at rest or continues to move at a constant velocity, unless acted upon by a force.
- **Second law**: In an inertial reference frame, the vector sum of the forces F on an object is equal to the mass m of that object multiplied by the acceleration a of the object:

$$F = ma.$$

- **Third law**: When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

[11] A historically (and substantially) important example of a shift from definition to law, and perhaps back again, occurs in the early discussions of what it means to be computable. See "Not a Definition, a Natural Law" in Allyn Jackson's essay *Emil Post: Psychological Fidelity* published in the journal *Inference* `https://inference-review.com/article/psychological-fidelity`

These are the starting laws, that launched that extraordinarily exact science—classical mechanics.

A. Kant's metaphysical counterpart to Newton's laws. Almost exactly a century later (1786) Kant published a treatise *Metaphysical Foundations of Natural Science*[12] as an attempt to revisit Newton's laws from a metaphysical perspective—and in a wildly idiosyncratic way. In contrast to Newton's very often-quoted "Hypotheses non fingo," (I make no—metaphysical—hypotheses) Kant is not shy of making them! For example, Here's Kant's definition of **matter**:

> **Matter** is whatever is movable and can be an object of experience.

The first definition in his first chapter is:

> **Definition 1:** I call something material if and only if it is movable in space. Any space that is movable is what we call material or **relative space**. What we think of as the space in which all motion occurs—space that is therefore absolutely immovable—is called pure space or **absolute space**.

Starting with this definition—to express it anachronistically—Kant is working out the issue of dependence on frame of reference—in particular the idea that motion is a relative notion. He proclaims as a *Principle:*

> Every motion that could be an object of experience can be viewed either as
> - the motion of a body in a space that is at rest or as
> - the rest of a body in a space that is moving in the opposite direction with equal speed. Its a free choice.

(This might be a point for discussion in our seminar, since, here and at other places in his essay, Kant is formulating something that—in modern language—might be phrased as the question of whether a *coordinate-free language* is possible for the framing of physical laws.)

Regarding gravitational attraction, he offers this description, but labelled as propositions:

[12] http://www.earlymoderntexts.com/assets/pdfs/kant1786.pdf

> **Proposition 7:** The attraction that is essential to all matter is an unmediated action through empty space of one portion of matter on another.

> **Proposition 8:** The basic attractive force, on which the very possibility of matter depends, reaches out directly from every part of the universe to every other part, to infinity.

And eventually, Kant formulates his version of the laws:

- **First law of mechanics:** Through all changes of corporeal Nature, the over-all amount of matter remains the same— neither increased nor lessened.
- **Second law of mechanics:** Every change in matter has an external cause. (Every motionless body remains at rest, and every moving body continues to move in the same direction at the same speed, unless an external cause compels it to change.)
- **Third mechanical law:** In all communication of motion, action and reaction are always equal to one another.

(A further possible topic for discussion might be whether Kant succeeds or fails in giving Newton's laws a metaphysical underpinning—or in elucidating the underlying issues. Or: what is Kant actually trying to get at?)

B. Ernst Mach's reconfiguration of Newton's laws. Going forward yet another century, there is Mach's retake. Most striking is his view of Newton's second Law. Simply formulated above, it was that the *law* $F = ma$ might be viewed as defining the concept of mass: the quantity m is, in fact *defined by its appearance as a constant* in such a law. Specifically, this second law is the following assertion:

> Appropriately understood —relative to any specific body (conceived of as "point-mass") the 'ratio' F/a is constant; and this constant is *defined* to be the mass of this body.

This turns the tables, a bit, on the second law, focusing on its role as a definition of one of the three components of this Newtonian system. Indeed Mach had a few variant formulations of this definition: one of them in terms of a little drama involving a closed system consisting of two bodies A and B doing whatever they are doing to each other (repelling or attracting) and as a result having accelerations a_A and a_B at which point (making implicit use of Newton's *Third law*) one can *define the ratio of the masses* of bodies A and B to be:

$$m_A/m_B = -a_B/a_A,$$

which has the sdded virtue of not ever dealing explicitly with the notion of Force[13]

C. D'Alembert's revision. More than a century earlier than Mach, there appeared a different rewriting of Newton's second law known (now) as D'Alembert's Principle. This principle seems to have been created by a shockingly simple—you might think:—trick. But, in fact, it represents an extremely important change of viewpoint related to (and perhaps inspiring) an entire genre of *conservation laws* and *stationary principles*. Here is D'Alembert's simple idea. Write Newton's second law, $F = ma$, in this (clearly equivalent) way:

$$F - ma \quad = \quad 0.$$

Dubbing "$-ma$" as a sort of fictional force (referred to as *inertial force*) you get that the equilibrium of the system is marked (in this somewhat semantic juggle) as the sum of all the forces on the system being: 0. A *conservation* of forces. This is the precursor of two other rewritings of Newton's laws—these going under the names of *Lagrange*, and *Hamilton*. (An interesting theme for a final paper would be a discussion of the way in which these formulations difffer from each other, and from Newton's original formulation.) Viewing Newton's second law (as D'Alembert did) as a 'conservation law' is in the spirit of other conservation laws, such as conservation of energy—which in a sense was more of a principle to be defended rather than a pure axiom, in that whenever, in some set-up, the 'conservation principle of energy' seemed to be violated, the physicists confronted by this seeming violation fashioned a new facet or form of energy to put into the equation so as to maintain conservation.

[13] For a very critical discussion of the merits of Mach's approach, see *There Is No Really Good Definition of Mass* by Eugene Hecht `http://physicsland.com/Physics10_files/Mass.pdf`

See also *About the definition of mass in (Machian) Classical Mechanics* by Marco Guerra and Antonio Sparzani (Foundations of Physics Letters, **7**, No. 1, 1994)

Part 4. Axioms for Set Theory, Algorithms, Gödel's Incompleteness Theorem

D. Readings:

(i) Martin Davis, *Gödel's incompleteness Theorem*, Notices of the AMS **53** (2006) 414-418
`http://www.ams.org/notices/200604/fea-davis.pdf`

(ii) Allyn Jackson, *Emil Post: Psychological fidelity,* Inference, **4** (2018)
`https://inference-review.com/article/psychological-fidelity`

(iii) Wikipedia on *Gödel's incompleteness Theorem* `https://en.wikipedia.org/wiki/G%C3%B6del%27s_incompleteness_theorems`

E. How rigorous will we be? The answer is: not rigorous at all, and in no way self-contained; but I shall only hint at the train of arguments—introducing a minimum of notation—but I'll try to usefully evoke the general sense of these arguments, and be as honest as I can be, in all this 'evoking.'.

14. Listable sets of integers

(synonyms: *recursively enumerable, computably enumerable, algorithmically enumerable*)

Let's start with some examples of sets that are easy to "list"

•
$$2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$$

•
$$2!, 3!, 4!, 5!, \ldots$$

Discuss what is meant by *easy.*

Generally, a subset $\mathcal{L} \subset \mathbb{Z}$ is called **listable**[14] if there exists a finite computer program whose output gives a sequence $\alpha_1, \alpha_2, \alpha_3, \dots$ of integers such that the set $\mathcal{L}$ is precisely this collection of numbers; i.e.,

$$\mathcal{L} = \{\alpha_1, \alpha_2, \alpha_3 \dots\}.$$

A computer algorithm that does job this will be called a computer algorithm that "lists $\mathcal{L}$."

Note, though, that–even if the computer spits out a "new" integer every second— the ordering in which the numbers in the computer's listing of $\mathcal{L}$ come may be very up and down in terms of size. Therefore if you suspect that a given number, say 3, is *not* in $\mathcal{L}$ and need to have a definite guarantee of the truth of your suspicion, well (if you are right!) running the computer algorithm for any finite length of time— with the helter-skelter sizes of numbers that come up will be of no help to you: if 3 does show up, it is—of course–therefore in $\mathcal{L}$; if it hasn't yet shown up, no matter how long the computer has run, this tells you nothing about whether it is or isn't in $\mathcal{L}$.

For example, (I'm taking an offhand random example) consider the set of numbers that are expressible as a sum of two sixth powers minus a sum of two sixth powers. Now the set $\mathcal{L}_o$ of such numbers is listable. There is a simple way of systematically listing all such numbers. Run systematically through all quadruples of whole numbers A, B, C, D organizing these quadruples by size in what is called 'diagonal ordering' and collecting the values $n := A^6 + B^6 - C^6 - D^6$, this constituting a list of the elements of $\mathcal{L}_o$.

What are the numbers n that are *not* in $\mathcal{L}_o$? I (personally) don't know[15] whether, say, 3 is or is not in $\mathcal{L}_o$.

Suppose you have a listable subset of positive numbers:

---

[14] A very readable introduction to a bit of this theory (and especially the historical and personal context in which it arose) is Allyn Jackson's essay; see D(ii) above. For a pretty readable intro to the notion of *listable set* see `https://en.wikipedia.org/wiki/Recursively_enumerable_set` noting that listable is called **recursively enumerable** there. For a hint about hierarchies of 'relative listability' take a quick look at `https://en.wikipedia.org/wiki/Post%27s_theorem`.

[15]. . . but I haven't thought much about how difficult it is to know. . .

$$\mathcal{L} \subset \mathbf{N} := \{1, 2, 3, \ldots\}.$$

By the **complement** of such a set $\mathcal{L}$ one means the set of all numbers in $\mathbf{N}$ that are *not in* $\mathcal{L}$. Call this complement $\mathcal{L}^{\perp}$. So the union of $\mathcal{L}$ and $\mathcal{L}^{\perp}$ is all of $\mathbf{N}$:

$$\mathcal{L} \cup \mathcal{L}^{\perp} \;=\; \mathbf{N}.$$

Lemma 14.1. Day and Night: *There is an algorithm to determine whether any number* $n = 1, 2, 3, \ldots$ *is contained in* $\mathcal{L}$ **if and only if** *both* $\mathcal{L}$ *and* $\mathcal{L}^{\perp}$ *are listable*[16].

Proof: If there is such an algorithm, go through the positive integers one by one, and for each integer n use the algorithm to determine whether it is or isn't contained in $\mathcal{L}$ and put that number n in the appropriate listing of $\mathcal{L}$ if the algorithm says it is, and in $\mathcal{L}^{\perp}$ if the algorithm says it isn't.

Going the other way, suppose that both $\mathcal{L}$ and $\mathcal{L}^{\perp}$ are listable. For any integer n spend your days listing $\mathcal{L}$ and your nights listing $\mathcal{L}^{\perp}$ and you are guaranteed to find n at some day or night; this gives us the algorithm that determines whether n is or isn't contained in $\mathcal{L}$.

15. Emil Post's Fundamental Discovery

A fundamental result of Cantor is that there are sets that are unlistable. This follows from the fact that the set of all *listable* sets is a countable set of sets because the set of all possible lists are countable... but the set of *all* sets is an uncountable set. So there has to exist some unlistable set.

But a fundamental discovery of Emil Post is that there exists a listable set $\mathcal{L}$ whose complement $\mathcal{L}^{\perp}$ is *unlistable*[17]. It follows (e.g., from Lemma 14.1) that there is no decision procedure to determine whether any given integer n is or is not contained in $\mathcal{L}$.

Much of what comes later are afterthoughts to—and elaborations of— this discovery!

[16] In the literature, if a set $\mathcal{L}$ has the property that both $\mathcal{L}$ and $\mathcal{L}^{\perp}$ are listable, then $\mathcal{L}$ is called **recursive**.

[17] I.e., following the previous footnote: (in the literature) the set $\mathcal{L}$ would be called *recursively enumerable* but not *recursive*.

16. Gödel's Incompleteness Theorem

Suppose we have a formal system $\mathcal{F}$ (e.g., as described in Section 9) that is **consistent**— i.e., is such that for no wff P is it true that P and $\neg P$ is provable in $\mathcal{F}$—and has a rich enough vocabulary to perform whatever it is required to do below[18]. We aren't being at all explicit here, so take this as a (perhaps overly relaxed) way of directly getting to the heart of the idea of the incompleteness theorem.

For our formal system $\mathcal{F}$ there is a language, we have the standard apparatus and clear rules of inference etc.as described in Section 9, so we can actually algorithmically list all well-formed-formulae. Do that for $m = 1, 2, 3, \ldots$, giving us a complete list

$$m \mapsto P_m \tag{16.1}$$

that runs through every well-formed formula.

Definition 2. *One says that a formal system is* **complete** *if for any proposition P formulated in the language of the system either the proposition P or its negation $\neg P$ is provable.*

Now take *any one* of Post's sets

$$\{1, 2, 3, \ldots\} \leftrightarrow \mathcal{L} \tag{16.2}$$

listable in the language of $\mathcal{F}$ such that the complement $\mathcal{L}^{\perp}$ is unlistable.

Theorem 16.3. *(Gödel) There is at least one positive integer ν for which neither the statement:*

$$P: \quad \nu \textit{ is not in } \mathcal{L}$$

nor its negation

$$\neg P: \quad \nu \textit{ is in } \mathcal{L}$$

is provable in the formal system $\mathcal{F}$.

Proof:

Euclidean geometry without the parallel postulate is incomplete, because some statements in the language (such as the parallel postulate itself) can not be proved from the remaining axioms. **Proof:** Working

[18] For example: we are requiring our formal systems to be **effectively axiomatized** (also called effectively generated) so that its set of theorems is a listable set. That is, there is a computer program that, in principle, could enumerate precisely the theorems of the system. The standard formal systems Peano arithmetic and Zermelo-Fraenkel set theory (ZFC) have that property.

systematically, take each positive integer n and spend your days and nights this way. For a fixed n here is the procedure, which I'll call $\mathbf{Proc}(n)$:

> **Each Day:** look at successive 100 entries in the list (16.2) of elements of $\mathcal{L}$ to check if n is among those entries. If, on some day, you find n in $\mathcal{L}$ your work is done. Just remember all that work.
>
> **Each Night:** examine successive 100 well-formed formulae in the list (16.1) to see if, for some P_m among those entries, *P_m is a proof that ν is not in $\mathcal{L}$*. If, on some night, you find that one of those well-form formulae P_m is a proof that ν is not in $\mathcal{L}$ your work is done. Just remember it.

OK, here are the possibilities:

(i) The 'procedure' $\mathbf{Proc}(n)$ terminates finitely for *every* positive integer n. That is, we have an algorithm that for all n determines in finite time whether n is or is not in $\mathcal{L}$.
(ii) There "is" a positive integer ν such that the 'procedure' $\mathbf{Proc}(\nu)$ never terminates.

Consider, first, (i) above: if (i) held it would give us a finite algorithm to list the elements of $\mathcal{L}^{\perp}$: for every n run systematically through your days and nights, throwing out the n's that show up in daytime but list the n's for which a proof P_m (that $n \in \mathcal{L}^{\perp}$) has been found in those nighttimes. This, in effect, gives an algorithmic listing of $\mathcal{L}^{\perp}$—*contrary to assumptions*. So, (i) cannot occur.

This leaves (ii). Note that such a number ν cannot be a member of $\mathcal{L}$, for it if were, it would be found some day. So

- ν *is* a member of $\mathcal{L}^{\perp}$; i.e., the statement *ν is in $\mathcal{L}$* is not provable; and:
- there's also no proof—in the formal system $\mathcal{F}$—of the negation of this—i.e., of the statement *ν is* **not** *in $\mathcal{L}$*—since if there were, the procedure $\mathbf{Proc}(\nu)$ would terminate.

QED

Questions:

(i) Why did I put quotation-marks in the "is" in the statement of (ii) above?
(ii) How does this discussion change when I change the formal system $\mathcal{F}$ (within which we are working)?

17. A Diophantine (synonym: 'arithmetic') formulation: the result of Matiyasevich-Robinson-Davis-Putnam: a counter-statement to Hilbert's Tenth Problem.

Theorem 17.1. *(MRDP) For any listable set of positive integers $\mathcal{L}$ there is a polynomial $p(t, x_1, x_2, \ldots, x_d)$ in some finite number $(d+1)$ of variables with integer coefficients such that n is in $\mathcal{L}$ if and only if there are integers $(a_1, a_2, \ldots, a_d)$ such that*

$$p(n, a_1, a_2, \ldots, a_d) = 0. \tag{17.2}$$

Corollary 17.3. *(Unsolvability: a counter-statement to Hilbert's Tenth Problem) there is a polynomial $p(t, x_1, x_2, \ldots, x_d)$ with integer coefficients for which there is no algorithm to determine for n running through all positive integers whether $p(n, x_1, x_2, \ldots, x_d)$ has a solution (i.e., integers $(a_1, a_2, \ldots, a_d)$ satisfying Equation 17.2).*

Proof of Corollary 17.3 (given Theorem 17.1): Just take any listable set $\mathcal{L}$ for which $\mathcal{L}^{\perp}$ is unlistable, use Theorem 17.1 to find the corresponding polynomial, and interpret what this means.

Unfortunately too brief discussion—but at least mention of these topics:

(i) Various ways of thinking about Gödel's Incompleteness Theorem. E.g., "independence."
(ii) Peano Arithmetic, ZFC.
(iii) Gödel's view of Set Theory.
(iv) Current 'programs': Harvey Friedman; Hugh Woodin.

18. What is a set?

I'm not sure we know (definitively) yet. It is a pretty lean mathematical object, evoked—if not captured—by the simple phrase *a collection of things*. Nevertheless *sets* provide the substrate for such a wide variety of mathematical objects that an axiom system that 'models' set theory is clearly of foundational importance in mathematics.

That the axiom system for Set Theory is free of contradictions (i.e., is 'consistent') is, of course, necessary; and we should take particular care to achieve consistency, especially with Russell's paradox as a cautionary tale.

The most common such axiom system, *Zermelo-Fraenkel Set theory* referred to as ZF—or if one adds *the axiom of choice* to its list of axioms, one calls it ZFC—was proposed in the early twentieth century by Ernst Zermelo and Abraham Fraenkel.

Instead of presenting this axiom system as a 'formulated thing,' it might be more engaging if we discuss it, building it (or something close to it) up, by stages in conversation. The most dramatic procedure for such building-up is due to Von Neumann.

We allow our *set theory* to have the standard first-order logic as semantics so we can use, for example, the standard quantifications $\forall$ and $\exists$, etc. .

Since we are trying to capture the notion of *set* as a formal entity having the intuitive meaning of *collection of things*, the most economical thing one might do—and Z-F does this!—is to

- have just one type of object in our vocabulary (these objects are either to be thought of as *sets* or *objects* that are members of some set or of some sets; or these objects are both sets in themselves and members of other sets: all this depends on their properties in connection with the unique relation ($\in$) that will be introduced in the bullet below; these objects will be designated by some letter (e.g., $x, y, \dots$)

and

- have just one formal relation (*membership*, denoted $\in$). So, for x, y in our discourse, it might be the case that $x \in y$—namely, x is a member of the set y. It might also be the case that $x \in y \in z$ (i.e., y is a member of the set z but is also a set in its own right, containing x as a member).

A. The ambiguity of "and so on." Building sets; as von Neumann did. Start with the set containing no element, the *empty set* $\{\ \}$ which we'll call $\varnothing$. We then—ridiculous as this may seem—consider the set containing only one element, the empty set: $\{\varnothing\}$. Well then, we can imagine keeping going: form the set containing the two elements:

$$\{\ \varnothing, \{\varnothing\}\ \},$$

the set containing the three elements:

$$\{\ \varnothing, \{\varnothing\}\ \{\ \varnothing, \{\varnothing\}\ \},$$

and so on...

Namely, this procedure proposes to construct an unlimited collection of sets (from nothing). In fact even the elements of these sets are, curiously, themselves sets. So *all the objects* of this discourse launched by von Neumann are themselves sets (!) Can we produce a system of axioms that

- formalizes the construction that von Neumann has proposed (so every object is a set) and
- provides a formal architecture reasonable for Set Theory?

The axiomatic system ZF consists of eight axioms: and if one throws in the very tricky Axiom of Choice as ninth axiom one calls the system ZFC.

B. Equality and 'Extensionality'. *Discuss "extension versus intention"*

We want two objects a, b of our discourse to be regarded as *equal* if—in English—every element of the set a is equal to an element of the set b and vice versa; *and also* every set containing a as an element also contains b as an element and vice versa. If you are worried about this looking suspiciously like a circular definition, in that the word *equal* appears in the formulation—how about:

$$\forall z[z \in x \Leftrightarrow z \in y] \wedge \forall w[x \in w \Leftrightarrow y \in w].$$

Now the **Axiom of Extensionality** can be formulated as

$$\forall x \forall y[\forall z(z \in x \Leftrightarrow z \in y) \Rightarrow \forall w(x \in w \Leftrightarrow y \in w)],$$

which says that if x and y have the same elements, then they belong to the same sets; i.e., they're equal.

Axioms that produce sets from other sets

C. Definition: A set z is a **subset** of a set x if and only if every element of z is also an element of x:

$$(z \subseteq x) \Leftrightarrow (\forall q(q \in z \Rightarrow q \in x)).$$

D. The Axiom of Specification: An axiom guaranteeing that we may create a subset of a given set z by imposing conditions on the elements of z.

If z is a set and $\phi(x)$ is some formula imposing a condition on the variable x then
$$A := \{x \in z : \phi(x)\}$$
is a set too.

Note that a formula $\phi(x)$ *alone* is not enough to produce a set (one of the troublesome features of various ontological proofs of the existence of God): this axiom requires you to stipulate a set z from which you want to cut out a piece (as subset) by imposing some predicate as a condition.

E. The Axiom of Power Sets: An axiom saying (essentially) that the 'collection' of all subsets of a given set is again itself a set (in its own right).

The Axiom of Power Set specifically states that for any set x, there is a set y that contains every subset of x:

$$\forall x \exists y \forall z [z \subseteq x \Rightarrow z \in y].$$

The axiom schema of specification can then be used to define the power set P(x) as the subset of such a y containing the subsets of x exactly:

$$P(x) = \{z \in y : z \subseteq x\}.$$

F. Axiom of pairing: If x and y are sets, then there exists a set B which contains x and y as elements.

$$\forall x \forall y \exists B (x \in B \wedge y \in B).$$
(The axiom schema of specification can be used to reduce this to a set with exactly these two elements.) Note that taking $x = y$ we then get that if x is any set, the singleton $\{x\}$ (set containing x as its unique member) is again a set.

G. Axiom of union. The union over the elements of a set exists.
The axiom of union states that for any set of sets

$$\mathcal{F}$$

there is a set C containing every element that is a member of some member of $\mathcal{F}$.

Letting $S(w)$ abbreviate $w \cup \{w\}$ where w is some set, we get—from the Axioms of pairing and union that $S(w)$ is indeed a set. So, for example, assuming that von Neumann's $\varnothing$ is assumed to exist as a set in our theory, each of the following creatures in von Neumann's list also are sets in our theory:

$$\varnothing, \quad \{\ \varnothing, \{\varnothing\}\ \}, \quad \{\ \varnothing, \{\varnothing\}\ \{\ \varnothing, \{\varnothing\}\ \}, \quad \text{etc.}$$

Note that it would be perfectly consistent with the collection of axioms discussed so far to have a model of set theory where every set is finite. In fact, there might be no sets at all in our model, so ...

H. Axiom of Infinity. Recall that $S(w) := w \cup \{w\}$ where w is a set. The Axiom of Infinity states that there exists a set X such that the empty set $\varnothing$ is a member of X and, whenever a set y is a member of X, then $S(y)$ is also a member of X. In effect, we are requiring that the ('infinite') union of the sets in von Neumann's list is *also a set in our theory.*

$$\exists X\, [\varnothing \in X \wedge \forall y (y \in X \Rightarrow S(y) \in X)]\,.$$

Now it would be good to know that X has infinitely many members (this issue is hinted at in the scare-quotes around the word "infinite' above). We need that the sets listed in the definition of X are all different, because if two of these sets are the same, the sequence will loop around in a finite cycle of sets. A Russellian quandary.

The *axiom of regularity* below is a clever way of establishing a basic property of the relation $\in$ and preventing loops from happening:

I. Axiom of regularity. We include as axiom the requirement that every non-empty set x contains a member y such that x and y are disjoint sets.

$$\forall x\, (x \neq \varnothing \rightarrow \exists y \in x\, (y \cap x = \varnothing)).$$

The remaining one (or two) axioms

The above evocation of the first 7 of the 8 axioms of ZF is meant to spark a discussion—I hope there's time for such discussion in our last sessions. We have not discussed the 8-th axiom **The Axiom of Replacement** which asserts that images of (appropriately) definable functions from sets to set, $f : S \to T$, are again sets; nor the final axiom **The Axiom of Choice** (that turns ZF to ZFC). But I hope we can talk a bit about the impact of Gödel's Incompleteness on the project of setting up a formal system that is comprehensive enough to be a 'foundation' for mathematical practice. And on the connection with higher cardinalities. And... the important issue of mathematical induction.