Notes for our seminar: Objectivity and Subjectivity

B.M.

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Contents

Part I

A preview and overview of the topics we may cover

1 The nature of our seminar-course

Phil273O **(Objectivity and Subjectivity)** will be the fourth seminar-course I've taught with Amartya Sen and Eric Maskin. Here are some directions our seminar *may* take; we will pay particular attention to topics that connect specifically with the experience, and interests, of the participants in our seminar.

1. **Objective**

   The noun *object*, the adjective *objective*, and the notion *objectivity* capture quite a range of thought. And there are many ways to think about, and to use them.

   To begin crudely, the adjective *objective* is sometimes used in a 'via negativa' sort of way; meaning: *un*-biased; that is: you label your assertion, or viewpoint, *objective* as a signal that despite the fact that one might worry that that viewpoint, or judgment, is tainted with the prejudices of complicating *subjective* or partial sentiments, you feel that it is *not*.

   A (putative) *objective view* would then offer a take on the matter... with no hint of subjective bias. The phrase *objective view* itself sometimes comes with a moral tone.

   It may be challenging to go on to give a more elaborate explanation of this usage. But the curious assumption here is that one can view something without actually taking a point of view; i.e., impartially, and either coordinate-free, or at least in an invariant way, independent of frame of reference—like the invariant formulations of Maxwell's Laws[1]

[1] But we also have Maxwell's comment—in his essay *Analogies in Nature* [5]—namely:

> "the only laws of matter are those which our minds must fabricate, and the only laws of mind are fabricated for it by matter"

(of Electromagnetism) or Einstein's Special Relativity.

Slightly different, and expressed in a somewhat more formal setting, is Donald Davidson's take on objectivity in his book: *Subjective, Intersubjective, Objective* [7].

> Thought, propositional thought, is *objective* in the sense that it has a content which is true or false independent (with rare exceptions) of the existence of the thought or the thinker.

As Lorraine Daston and Peter Galison mention in their article "the Image of Objectivity" [6] and their book "Objectivity" [5] 'the word "objectivity" has a somersault history.' In the writings of Duns Scotus and William of Ockham the word appears in adjectival form rather than substantive. Moreover:

> ... the terms [objective/subjective] meant almost precisely the opposite of what they mean today. "Objective" referred to things as they are presented to consciousness, whereas "subjective" referred to things in themselves. (See page 29 of [5][2])

Commentator's on Acquinas try to explain his notion of *essence* (Essentia) —cf. [2]— as a way of bridging the distinction between *formal concept* and what they call *objective concept*[3] E.g., Thomas Cajetan (1469-1534) writes, in connection with Acquinas's thought:

> ... For example, the formal concept of a lion is that representation where the possible intellect forms of a leonine quiddity when we want to know it; the objective concept of the same thing is the leonine nature itself, represented and known.

---

[2] And compare this to Kant's 'Copernican revolution;' see footnote 4 below.

Daston and Galison give a historical account of the evolution of the notion of *scientific objectivity* and as it applies to the daily practice of science. More specifically, they focus on the visual images in scientific atlases (from atlases of flora in the eighteenth century to more modern records—ranging over a host of different 'objects of scientific enquiry'). They see a progression—an evolution—of ways of producing, and ways of understanding, the images presented as 'science,' and they label three phases in the manner of production and choice of those presentations; in chronological order: *truth-to-nature*, *mechanical objectivity*, and *trained judgment*. In effect, they offer a history of the changing attitudes toward objectivity in science.

[3] Although, I think the phrase *substantive concept* might convey more accurately the sense of Acquinas's text than *objective concept*.

See page 326 of [11] and/or Page 103 of [13].

Regarding the noun 'object,'

- the arresting turn of thought in Kant's *Critique of Pure Reason* where *object* seems to be viewed as *ob- ject*—a thing that is thrown out' by, in effect, the subject and dressed in clothes (such as "space and time") so that it can be properly re-presented (back to that very subject) as something that can be thought about. This revision (of the ordinary sense of meaning of *object*—and consequently also of the very notion of *idea*) is described by Kant (in the Preface to the second Edition of the Critique) as a 'Copernican Revolution.'[4] Kant paints a picture regarding *ideas as objects of thought* quite different from that offered by any of his predecessors;

e.g., (and here we'll ignore chronological order, but nevertheless begin with):

- the transcendental nature[5] of Plato's *eide*—i.e., the main ingredient in what is often referred to as his *theory of forms,*

or

- the axiomatic turn of mind of Spinoza, as reflected in his formulating "A true idea must agree with its object" as an Axiom in his *Ethics*,

or, taking a somewhat different direction:

- Descartes offers the proposition that a greater measure of *objective*

[4] From Kant's [8] (Bxvi):

> Up to now it has been assumed that all our cognition must conform to the objects; but all attempts to find out something about them a priori through concepts that would extend our cognition have, on this presupposition, come to nothing. Hence let us once try whether we do not get farther with the problems of metaphysics by assuming that the objects must conform to our cognition, which would agree better with the requested possibility of an a priori cognition of them, which is to establish something about objects before they are given to us. This would be just like the first thoughts of Copernicus, who, when he did not make good progress in the explanation of the celestial motions if he assumed that the entire celestial host revolves around the observer, tried to see if he might not have greater success if he made the observer revolve and left the stars at rest. Now in metaphysics we can try in a similar way regarding the intuition of objects. If intuition has to conform to the constitution of the objects, then I do not see how we can know anything of them a priori; but if the object (as an object of the senses) conforms to the constitution of our faculty of intuition, then I can very well represent this possibility to myself.

[5] The idea of a subject-less idea is... arresting.

reality ("realitatis objectivae") are conveyed by 'ideas' that represent *substances* (this occurs in a curious context in his *Méditations*-–whose subtitle is "First Philosophy"),

these attitudes regarding 'idea' being in contrast to

- the relaxed view of John Locke who comes out against the existence of "innate ideas" and for whom the notion *idea* is simply

  > "the best word to stand for whatever is the object of the understanding when a man thinks; I have used it to express whatever is meant by phantasm, notion, species, or whatever it is that the mind can be employed about in thinking."

and consider the notions of *idea* or *law* as applicable to the sciences, with the expectation that such formulations

- have *scientific objectivity*; i.e., if they are experimental findings, are deduced objectively; e.g., by *double-blind*—and/or *randomized controlled* experiments
- and have the aim of describing *Nature* (as something that might admit objective description).

The concept of **Nature**, as the substrate of everything, and as the prime target of objective description (of "reality")

- pervades ancient Greek thought ($\phi\upsilon\sigma\iota\varsigma$) but is, perhaps, most explicitly focused on in Aristotle's *Physics*.
- It is succinctly encapsulated by Lucretius's:

  All nature, then, as self-sustained, consists
  Of twain of things: of bodies and of void
  In which they're set, and where they're moved around.

  (this is in his *De Rerum Natura*).
- Francis Bacon brusquely sets forth the manner in which Nature should (not) be investigated,[6] in the Preface to his *Novum Organum*:

[6] For interesting discussion of Bacon's take on Nature and on Objectivity, see: [14] and [17].

> They who have presumed to dogmatize on nature, as on some well investigated subject, either from self-conceit or arrogance, and in the professorial style, have inflicted the greatest injury on philosophy and learning. For they have tended to stifle and interrupt inquiry exactly in proportion as they have prevailed in bringing others to their opinion: and their own activity has not counterbalanced the mischief they have occasioned by corrupting and destroying that of others.

– Wordsworth, in his poem *The Prelude*, conceives of Nature as an agent, and one to whom one might show gratitude:

  > This verse is dedicate to Nature's self
  > And things that teach as Nature teaches

  or to whom one might address in praise:

  > ...O Nature! Thou hast fed
  > My lofty speculations...

– In the era when the Theory of Relativity was freshly on the scene—and even before Quantum Mechanics had entered—one can see a slight shift of attitude toward the "agent" Nature. Henri Poincaré in his treatise *Science and Method* asks, rhetorically.

  > Is nature governed by caprice, or is harmony the reigning influence ? That is the question.

  and offers:

  > It is when science reveals this harmony that it becomes beautiful, and for that reason worthy of being cultivated.

  while Bertrand Russell, in the preface to that treatise takes a less grand, and somewhat pragmatic approach:

  > The conception of the "working hypothesis," provisional, approximate, and merely useful, has more and more pushed aside the comfortable eighteenth century conception of "laws

> of nature." Even the Newtonian dynamics, which for over two hundred years had seemed to embody a definite conquest, must now be regarded as doubtful, and as probably only a first rough sketch of the ways of matter.

and Poincaré emphasizes the inherently subjective choices necessary to be made in the enterprise of science as it explores nature:

> Trying to make science contain nature is like trying to make the part contain the whole.

Poincaré begins his treatise by noting:

> Tolstoi explains somewhere in his writings why, in his opinion, "Science for Science's sake " is an absurd conception. We cannot know all the facts, since they are practically infinite in number. We must make a selection; and that being so, can this selection be governed by the mere caprice of our curiosity? Is it not better to be guided by utility, by our practical, and more especially our moral, necessities?

– Nevertheless "Letting Nature speak for itself" as a label for "scientific objectivity" emphasizes a desire—idealized and unreachable as it may be. (See the discussion in [5].)

2. **Subjective**

As for the noun *subject*, the adjective *subjective*, and the notion *subjectivity*—these capture a similar range. Often only implicit is the presence (and nature) of the actual *subject* emanating from whom is the "subjective" viewpoint that is being examined. From

- Kant's striking concept: the *universal subjective*—a yin/yang combination of objective and subjective thought—that, according to Kant, is a fundamental ingredient of aesthetic judgments (see [9]). Namely, we are all equipped with an internal *universal subjective* temperament that—according to Kant— consists of a model (in our thoughts) of *all-of-humanity* (this 'model' may or not be an accurate portrayal of the sensibilities of all humanity; it doesn't matter). We necessarily invoke this model, in order to make aesthetic judgments–e.g.,"this song is beautiful"—by thinking (and referring to the model) that all humanity would/or/should concur with our judgment. This type of aesthetic judgment is on a different plane and is quite different from "simple likings," such as liking this particular ice cream cone;

to

- what one might call the *communal subjective*; that is, any resolution in a group enterprise where, after deliberation, a consensus is reached so that the community agrees that a certain statement should be taken as fact. For example it was announced that the Higgs Boson was discovered, of course, only after there was sufficient evidence, but—as I understand it—the conclusion that the evidence was actually sufficient was agreed upon by a vote taken among the physicists involved.

  This brand of fact is surely on the level of 'objective science,' and yet carries a tinge of subjectivity (communal agreement) that is on quite a different plane from, say, the Kantian sensus communis;

to

- the Bayesian view of probability, where the 'subjectivity' of one's prior assessment of probability gets incrementally 'educated" by the feedback loop of further data;

to

- the vast literature regarding 'the will'—including the conundra presented by the notions of free will and determinism. If anyone is interested in taking this up in a final paper project, we can offer an appropriate reading list: this topic certainly is within the span of our seminar-course.

3. **The contrast:** *objective* **versus** *subjective*

- It has a similar feel to the apposition: *knowledge* versus *opinion* as in the ancient literature (*episteme* versus *doxa*). A similar such dichotomy occurs in most of the later treatises regarding "human understanding." E.g., John Locke formulates it in Chapter XXI *Of the Division of the Sciences* in his *Essay Concerning Human Understanding*: "discovery of truth" (or what Locke elsewhere delineates as "the objects of understanding;") versus thoughts about "things in [the subject's] own power, which are his own actions, for the attainment of his own ends."[7]

- In Ethics, where the subject as agent—willful, responsible, innocent or culpable—plays a predominant role, much has been made of various 'objective anchors' that may be expected to interact with that subjective will. Kant's categorical imperative is an example of this. And here ([10]) is Kant emphasizing the *meta*-aspect of this imperative: it is not so much a rule, but rather one that is...sort of... quantified over $\forall$ rules:

  > I ought never to conduct myself except so that I could also will that my maxim become a universal law. Here it is mere lawfulness in general (without grounding it on any law determining certain actions) that serves the will as its principle, and also must so serve it, if duty is not to be everywhere an empty delusion and a chimerical concept; common human reason, indeed, agrees perfectly with this in its practical judgment, and has the principle just cited always before its eyes.

- In Mathematics, the issue (objective versus subjective) spans attitudes labelled *mathematical platonism, intuitionism, formalism.*
  - Mathematical platonism takes mathematical substance as having an essence independent of human thought; as being part of a prenoetic structure of the cosmos; and the aim of mathematics is to faithfully describe it.
  - Intuitionism, in its various forms, puts the spotlight on the manner in which mathematics is actually thought.[8]

[7]Locke goes on to talk of:

> "the signs the mind [that] makes use of both in the one and the other, and the right ordering of them, for its clearer information."

[8]There is quite a range of literature about this. Here's one interesting example: [3].

- (The Kantian take on this is marvelously subtle: not easily classifiable.)
- A formalist approach focusses specifically on the *language* of mathematics as holding the key to its meaning.

A good start for understanding all this is to take a look at the graphic novel *Logicomix: An epic search for truth* by Apostolos Doxiadis and Christos Papadimitriou. And David Hilbert's essay: *On the Infinite*.

What genre of items (of viewpoints, assertions, etc.) can or should fall under the categories: objective, or subjective, or neither? What do these terms serve? I.e., what would we lose if we simply erased it from our thoughts? What, possibly, might we gain (if we ignore these notions)? How were (and how are) they used? Abused?

These are suggestions for some themes for our seminar-course, and for some possible choices of directions to pursue in a final paper.

Except for the first ("introductory") session and the final ("wrapping up") session, each of our other sessions will be 'chaired' by one of the professors. To say that we each 'chair' a session means that we expect full involvement of students in discussions and also, at times, presentations. That is, besides a final paper for the course, we may request a (usually very short) presentation on the part of some volunteers.

- The substance of the final paper should be 'topical,' in the sense that it should be directly related to the discussions and reading that we have done.
- it should investigate some issue that you actually want to know about, or care deeply about;
- and (it would be great if it can) make use of your own expertise and experience.

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Part II

Readings and Notes for the sessions of Sept 11, Oct 2, Oct 30, and part of Dec 4

2 Readings for the September 11 session

1. Chapter 1, “First Person Authority” (Pages 1-14) Of [7].

2. Try—and it is a challenge—to make sense of Kant's notion of Universal Subjective as hinted at in the brief excerpt from Kant's *Critique of Judgment* ([9]) given in Section 2.1 below. Note that **(a)** the words *presentation* and *intuition* have, for Kant, specific technical meaning, and **(b)** there is something interestingly raw in (any of) Kant's formulations that tend (in my opinion) to be shined up and made less radical in many of the secondary accounts I've read.

3. Pages 81-84 in [6].

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[1] Davidson, D., *Subjective, Intersubjective, Objective*, Clarendon Press http://epistemh.pbworks.com/w/file/fetch/45787947/0198237529%20-%20Davidson,%20Donald%20(2001

[2] Daston, L., Galison, P., The Image of Objectivity, Representations, Vol. 40 Autumn, 1992; (pp. 81-128) http://cspeech.ucd.ie/Fred/docs/Galison.pdf

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2.1 Excerpt from [9]

> ... if a presentation by which an object is given is, in general, to become cognition, we need imagination to combine the manifold of intuition, and understanding to provide the unity of the concept uniting the [component] presentations. This state of free play of the cognitive powers, accompanying a presentation by which an object is given, must be universally communicable; for cognition, the determination of the object with which given presentations are to harmonize (in any subject whatever) is the only way of presenting that holds for everyone. But the way of presenting [which occurs in a judgment of taste is to have subjective universal communicability without presupposing a determinate concept; hence this subjective universal communicability can be nothing but that of the mental state in which we are when imagination and understanding are in free play (insofar as they harmonize with each other as required for cognition in general). For we are conscious that this subjective relation suitable for cognition in general must hold just as much for everyone, and hence be just as universally communicable, as any determinate cognition, since cognition always rests on that relation as its subjective condition. Now this merely subjective (aesthetic) judging of the object, or of the presentation by which it is given,

> precedes the pleasure in the object and is the basis of this pleasure, [a pleasure] in the harmony of the cognitive powers. But the universal subjective validity of this liking, the liking we connect with the presentation of the object we call beautiful, is based solely on the mentioned universality of the subjective conditions for judging objects. That the ability to communicate one's mental state, even if this is only the state of one's cognitive powers, carries a pleasure with it, could easily be established (empirically and psychologically) from man's natural propensity to sociability. But that would not suffice for our aim here. When we make a judgment of taste, the pleasure we feel is something we require from everyone else as necessary, just as when we call something beautiful, we had to regard beauty as a characteristic of the object, determined in it according to concepts, even though in fact, apart from a reference to the subject's feeling, beauty is nothing by itself. We must, however, postpone discussion of this question until we have answered another one, namely, whether and how aesthetic judgments are possible a priori.

3 Some notes for Sep 11: Subjectivity, Universal Subjectivity, Objectivity

What is Science? What is Mathematics? What is Music? What is Poetry?

It would seem that to begin to answer any of these (extremely naively wrought) questions you would have to understand, and deal with, the kind of objectivity (or subjectivity) that is possible, and appropriate to aim for, in the first two activities; and with the kind of extreme subjectivity that nevertheless goes along with universal human appeal in the second two.

To develop a vocabulary suitable, and helpful, to discuss those fundamental notions, 'objectivity/subjectivity,' and to use this vocabulary to get a better purchase on possible answers to these questions is one aim of our seminar.

3.1 Objective

The noun *object*, the adjective *objective*, and the notion *objectivity* capture quite a range of thought. And there are many ways to think about, and to use them.

1. **Objective** defined as a negative:

   To begin crudely, the adjective *objective* is sometimes used in a 'via negativa' sort of way; meaning: *un*-biased; that is: you label your assertion, or viewpoint, *objective* as a signal that despite the fact that one might worry that that viewpoint, or judgment, is tainted with the prejudices of complicating *subjective* or partial sentiments, you feel that it is *not*.

   A (putative) *objective view* would then offer a take on the matter... with no hint of subjective bias. The phrase *objective view* itself sometimes comes with a moral tone.

   It would be helpful to have a discussion in class about exactly this: the ethical and/or moral weight of—self-described—objective thought, objective knowledge, objective 'presentations.' The ethical formats of objectivity that scientists, and others—for example: journalists—are often professionally obliged to aim for:

   > [T]he belief in objectivity in journalism, as in other professions, is not just a claim about what kind of knowledge is reliable. It is also a moral philosophy, a declaration of what kind of thinking one should engage in, in making moral decisions.[9]

   Also:

   > Buffeted by controversy and powerful crosscurrents in society, these journalists [*in the period after WWI*] looked for a way forward for their profession. They invented objectivity as an ethical signpost in troubled times. [10]

[9] This is from [15] as quoted in Peter Galison's essay *The Journalist, the Scientist, and Objectivity* (Chapter 4 in [1]).

[10] This is from [16] also quoted in Galison's essay.

The *Society of Professional Journalists* adopted a specific "Code of Ethics" (cf. pages 65, 66 in [1]).

For Discussion: Given your particular background, and the field in which you are engaged, are there broad discussions formulating some version of 'objectivity' and the ethics involved—particular for that field?

2. **Objective** defined as a 'view from nowhere':

   It may be challenging to go on to give a more elaborate explanation of this usage. But the curious assumption here is that one can view something without actually taking a point of view; i.e., impartially, and either coordinate-free, or at least in an invariant way, independent of frame of reference—like the invariant formulations of Maxwell's Laws[11] (of Electromagnetism) or of Einstein's Special Relativity[12]. Later this semester Amartya will be discussing Thomas Nagel's book "The View from Nowhere" which investigates this.

3. **Objectivity as defined by appropriate practice**

   Journalists are expected to follow an appropriate protocol to establish (in effect, what is *defined to be*) the objectiveness of their reporting—this includes the 'inverted triangle' format of expansion of details having to do with the "WHO-WHAT-WHEN-WHERE of the thing reported on; it also includes appropriate checking and confirming the checks on 'facts,' a stab at evenhandedness in covering controversial issues; and, of course, no personal opinions interjected.

   For scientists, the basic outline structure of an experiment, or study, is something agreed upon by the community. Double-blind studies; randomized controlled experiments. Among many such guidelines there is the curious business of *p-value* which is supposed to give a sense of whether your data confirms the *null hypothesis* or not. The (somewhat arbitrary) convention is that if this p-value (which can live in a range

[11] But we also have Maxwell's comment—in his essay *Analogies in Nature* [5]—namely:

> "the only laws of matter are those which our minds must fabricate, and the only laws of mind are fabricated for it by matter"

[12] even though Einstein's laws—or at least all of them—would not be independent of ballet dancers who pirouette...

from 0 to 1) is less than $p = .05$ then your data is defined to be *statistically significant* contradicting the null hypothesis.

For discussion: 'Convened' notions of objective protocol in other domains (of study or practice)?

4. **Agents of authority for Objectivity**

Slightly different, and expressed in a somewhat more formal setting, is Donald Davidson's take on objectivity in his book: *Subjective, Intersubjective, Objective* [7].

> Thought, propositional thought, is *objective* in the sense that it has a content which is true or false independent (with rare exceptions) of the existence of the thought or the thinker.

And there are genres of thought where truth is not "independent of the thought of the thinker." Or independent of the identity of the thinker. Davidson focuses on the 'speaker' (referred to as *First Person*) and points out that "All propositional attitudes exhibit first person authority, but in various degrees and kinds."

We've discussed cases where the conventions of a community establish a formal status—some authority—to claims of objectivity. Davidson explores the issue of individual authority (e.g., "first person authority") and here we move to questions of levels of subjective truths—i.e., starting truths about beliefs or desires:

> When a speaker avers that he has a belief, hope, desire or intention, there is a presumption that he is not mistaken, a presumption that does not attach to his ascriptions of similar mental states to others. Why should there be this asymmetry between attributions of attitudes to our present selves and attributions of the same attitudes to other selves? What accounts for the authority accorded first person present tense claims of this sort, and denied second or third person claims?

Davidson continues:

Belief and desire are relatively clear and simple examples, while intention, perception, memory, and knowledge are in one way or another more complex. Thus in evaluating someones claim to have noticed that the house is on fire, there are at least three things to consider: whether the house is on fire, whether the speaker believes the house is on fire, and how the fire caused the belief. With respect to the first, the speaker has no special authority; with respect to the second, he does; and with respect to the third, responsibility is mixed and complex.

5. The tangle of Objectivity and Subjectivity

As Lorraine Daston and Peter Galison mention in their article "the Image of Objectivity" [6] and their book "Objectivity" [5] 'the word "objectivity" has a somersault history.' In the writings of Duns Scotus and William of Ockham the word appears in adjectival form rather than substantive. Moreover:

> . . . the terms [objective/subjective] meant almost precisely the opposite of what they mean today. "Objective" referred to things as they are presented to consciousness, whereas "subjective" referred to things in themselves. (See page 29 of [5].)

> And compare this to Kant's 'Copernican revolution;' see footnote 6 below.

> Daston and Galison give a historical account of the evolution of the notion of *scientific objectivity* and as it applies to the daily practice of science. More specifically, they focus on the visual images in scientific atlases (from atlases of flora in the eighteenth century to more modern records—ranging over a host of different 'objects of scientific enquiry'). They see a progression—an evolution—of ways of producing, and ways of understanding, the images presented as 'science,' and they label three phases in the manner of production and choice of those presentations; in chronological order: *truth-to-nature*, *mechanical objectivity*, and *trained judgment*. In effect, they offer a history of the changing attitudes toward objectivity in science.

Commentator's on Acquinas try to explain his notion of *essence* (Essentia) —cf. [2]— as a way of bridging the distinction between *formal concept* and what they call *objective concept*[13] E.g., Thomas Cajetan (1469-1534) writes, in connection with Acquinas's thought:

> ... For example, the formal concept of a lion is that representation where the possible intellect forms of a leonine quiddity when we want to know it; the objective concept of the same thing is the leonine nature itself, represented and known.

See page 326 of [11] and/or Page 103 of [13].

Regarding the noun 'object,'

- the arresting turn of thought in Kant's *Critique of Pure Reason* where *object* seems to be viewed as *ob- ject*—a thing that is thrown out' by, in effect, the subject and dressed in clothes (such as "space and time") so that it can be properly re-presented (back to that very subject) as something that can be thought about. This revision (of the ordinary sense of meaning of *object*—and consequently also of the very notion of *idea*) is described by Kant (in the Preface to the second Edition of the Critique) as a 'Copernican Revolution.'[14] Kant paints a picture regarding *ideas as objects of thought* quite different from that offered by any of his predecessors;

e.g., (and here we'll ignore chronological order, but nevertheless begin with):

[13] Although, I think the phrase *substantive concept* might convey more accurately the sense of Acquinas's text than *objective concept*.

[14] From Kant's [8] (Bxvi):

> Up to now it has been assumed that all our cognition must conform to the objects; but all attempts to find out something about them a priori through concepts that would extend our cognition have, on this presupposition, come to nothing. Hence let us once try whether we do not get farther with the problems of metaphysics by assuming that the objects must conform to our cognition, which would agree better with the requested possibility of an a priori cognition of them, which is to establish something about objects before they are given to us. This would be just like the first thoughts of Copernicus, who, when he did not make good progress in the explanation of the celestial motions if he assumed that the entire celestial host revolves around the observer, tried to see if he might not have greater success if he made the observer revolve and left the stars at rest. Now in metaphysics we can try in a similar way regarding the intuition of objects. If intuition has to conform to the constitution of the objects, then I do not see how we can know anything of them a priori; but if the object (as an object of the senses) conforms to the constitution of our faculty of intuition, then I can very well represent this possibility to myself.

- the transcendental nature[15] of Plato's *eide*—i.e., the main ingredient in what is often referred to as his *theory of forms,*

or

- the axiomatic turn of mind of Spinoza, as reflected in his formulating "A true idea must agree with its object" as an Axiom in his *Ethics*,

or, taking a somewhat different direction:

- Descartes offers the proposition that a greater measure of *objective reality* ("realitatis objectivae") are conveyed by 'ideas' that represent *substances* (this occurs in a curious context in his *Méditations*-–whose subtitle is "First Philosophy"),

these attitudes regarding 'idea' being in contrast to

- the relaxed view of John Locke who comes out against the existence of "innate ideas" and for whom the notion *idea* is simply

> "the best word to stand for whatever is the object of the understanding when a man thinks; I have used it to express whatever is meant by phantasm, notion, species, or whatever it is that the mind can be employed about in thinking."

6. Subjective

As for the noun *subject*, the adjective *subjective*, and the notion *subjectivity*—these capture a similar range. Often only implicit is the presence (and nature) of the actual *subject* emanating from whom is the "subjective" viewpoint that is being examined. From

- Kant's striking concept: the *universal subjective*—a yin/yang combination of objective and subjective thought—that, according to Kant, is a fundamental ingredient of aesthetic judgments (see [9]). Namely, we are all equipped with an internal *universal subjective* temperament that—according to Kant— consists of a model (in our thoughts) of *all-of-humanity* (this 'model' may or not be an accurate portrayal

[15] The idea of a subject-less idea is... arresting.

of the sensibilities of all humanity; it doesn't matter). We necessarily invoke this model, in order to make aesthetic judgments–e.g.,"this song is beautiful"—by thinking (and referring to the model) that all humanity would/or/should concur with our judgment. This type of aesthetic judgment is on a different plane and is quite different from "simple likings," such as liking this particular ice cream cone;

to

- what one might call the *communal subjective*; that is, any resolution in a group enterprise where, after deliberation, a consensus is reached so that the community agrees that a certain statement should be taken as fact. For example it was announced that the Higgs Boson was discovered, of course, only after there was sufficient evidence, but—as I understand it—the conclusion that the evidence was actually sufficient was agreed upon by a vote taken among the physicists involved.

  This brand of fact is surely on the level of 'objective science,' and yet carries a tinge of subjectivity (communal agreement) that is on quite a different plane from, say, the Kantian sensus communis;

to

- the Bayesian view of probability, where the 'subjectivity' of one's prior assessment of probability gets incrementally 'educated" by the feedback loop of further data;

to

- the vast literature regarding 'the will'—including the conundra presented by the notions of free will and determinism. If anyone is interested in taking this up in a final paper project, we can offer an appropriate reading list: this topic certainly is within the span of our seminar-course.

7. **The contrast:** *objective* **versus** *subjective*

   - It has a similar feel to the apposition: *knowledge* versus *opinion* as in the ancient literature (*episteme* versus *doxa*). A similar such dichotomy occurs in most of the later treatises regarding "human understanding." E.g., John Locke formulates it in Chapter XXI *Of the Division of the Sciences* in his *Essay Concerning Human Understanding*: "discovery of truth" (or what Locke elsewhere delineates as "the objects of understanding;") versus thoughts about "things in [the subject's] own power, which are his own actions, for the attainment of his own ends."[16]

- In Ethics, where the subject as agent—willful, responsible, innocent or culpable—plays a predominant role, much has been made of various 'objective anchors' that may be expected to interact with that subjective will. Kant's categorical imperative is an example of this. And here ([10]) is Kant emphasizing the *meta*-aspect of this imperative: it is not so much a rule, but rather one that is…sort of… quantified over ∀ rules:

  > I ought never to conduct myself except so that I could also will that my maxim become a universal law. Here it is mere lawfulness in general (without grounding it on any law determining certain actions) that serves the will as its principle, and also must so serve it, if duty is not to be everywhere an empty delusion and a chimerical concept; common human reason, indeed, agrees perfectly with this in its practical judgment, and has the principle just cited always before its eyes.

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> "the signs the mind [that] makes use of both in the one and the other, and the right ordering of them, for its clearer information."

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4 Readings for October 2:

(Issues of Objectivity and Subjectivity in Mathematics)

1. **Mathematical Platonism and its Opposites**

   This consists of five pages of notes I once wrote regarding the 'objectiveness' or 'subjectiveness' of mathematics [5]. I've put it on the web-page of our course. (The references [6], [7], and [8] listed in the bibliography below are not *at all* meant to be primary readings for our session, but just background material containing discussions about the nature of mathematics.)

2. Hilbert's **"On the infinite"**

   An address delivered by David Hilbert in which he introduced (for the first time! the notion of what is in effect) the concept of *formal system* as providing foundational languages for mathematics and formats for proving propositions in mathematics. For the full text go to [2].

Below is a six-page excerpt of it that may be useful for our discussions. (As the title suggests, the essay tries to understand various takes on the idea of the infinite.) But the gist, and main influencing force, of this essay is to highlight the importance of the notion: *formal system* (of axioms and rules of inference) and the importance of it being free of contradictions. If it is free of contradictions (i.e., if it is "consistent) then propositions proved via such a formal system have validity, even if the formal system makes use of certain ideal elements that have no meaning (i.e., think of the 16th century attitude towards imaginary numbers that are used in solving equations). The drama of this essay is that Hilbert claims to have proved (in the parts of the essay that have, for good reason, not been saved!) that—in effect—a formal system in which one can do arithmetic is free of contradictions. His 'proof,' must have been wrong, since... this is exactly what Gödel later showed to be impossible to prove (at least within the given formal system).

Excerpt from Hilbert's **On the Infinite**

Delivered June 4, 1925, before a congress of the Westphalian Mathematical Society in Munster, in honor of Karl Weierstrass. Translated by Erna Putnam and Gerald J. Massey from Mathematische Annalen (Berlin) vol. 95 (1926), pp. 161-90.

As a result of his penetrating critique, Weierstrass has provided a solid foundation for mathematical analysis. By elucidating many notions, in particular those of minimum, function, and differential quotient, he removed the defects which were still found in the infinitesimal calculus, rid it of all confused notions about the infinitesimal, and thereby completely removed the difficulties which stem from that concept. If in analysis today there is complete agreement and certitude in employing the deductive methods which are based on the concepts of irrational number and limit, and if in even the most complex questions of the theory of differential and integral equations, notwithstanding the use of the most ingenious and varied combinations of the different kinds of limits, there nevertheless is unanimity with respect to the results obtained, then this happy state of affairs is due primarily to Weierstrass's scientific work. And yet in spite of the foundation Weierstrass has provided for the infinitesimal calculus, disputes about the foundations of analysis still go on. These disputes have not terminated because the meaning of the infinite, as that concept is used in mathematics, has never been completely clarified. Weierstrass's analysis did indeed eliminate the infinitely large and the infinitely small by reducing statements about them to [statements about] relations between finite magnitudes. Nevertheless the infinite still appears in the infinite numerical series which defines the real numbers and in the concept of the real number system which is thought of as a completed totality existing all at once. In his foundation for analysis, Weierstrass accepted unreservedly and used repeatedly those forms of logical deduction in which the concept of the infinite comes into play, as when one treats of all real numbers with a certain property or when one argues that there exist real numbers with a certain property. Hence the infinite can reappear in another guise in Weierstrass's theory and thus escape the precision imposed by his critique. It is, therefore, the problem of the infinite in the sense just indicated which we need to resolve once and for all. Just as in the limit processes of the infinitesimal calculus, the infinite in the sense of the infinitely large and the infinitely small proved to be merely a figure of speech, so too we must realize that the infinite in the sense of an infinite totality, where we still find it used in deductive methods, is an illusion. Just as operations with the infinitely small were replaced by operations with the finite which yielded exactly the same results and led to exactly the same elegant formal relationships, so in general must deductive methods based on the infinite be replaced by finite procedures which yield exactly the same results; i.e., which make possible the same chains of proofs and the same methods of getting formulas and theorems. The goal of my theory is to establish once and for all the certitude of mathematical methods. This is a task which was not accomplished even during the critical period of the infinitesimal calculus. This theory should thus complete what Weierstrass hoped to achieve by his foundation for analysis and toward the accomplishment of which he has taken a necessary and important step. But a still more general perspective is relevant for clarifying the concept of the infinite. A careful reader will find that the literature of mathematics is glutted with inanities and absurdities which have had their source in the infinite. For example, we find writers insisting, as though it were a restrictive condition, that in rigorous mathematics only a finite number of deductions are admissible in a proof as if someone had succeeded in making an infinite number of them. Also old objections which we supposed long abandoned still reappear in different forms. For example, the following recently appeared: Although it may be possible to introduce a concept without risk, i.e., without getting contradictions, and even though one can prove that its introduction causes no contradictions to arise, still the introduction of the concept is not thereby justified. Is not this exactly the same objection which was once brought against complex-imaginary numbers when it was said: "True, their use doesn't lead to contradictions. Nevertheless their introduction is unwarranted, for imaginary magnitudes do not exist"? If, apart from proving consistency, the question of the justification of a measure is to have any meaning, it can consist only in ascertaining whether the measure is accompanied by commensurate success. Such success is in fact essential, for in mathematics as elsewhere success is the supreme court to whose decisions everyone submits. As some people see ghosts, another writer seems to see contradictions even where no statements whatsoever have been made, viz., in the concrete world of sensation, the "consistent functioning" of which he takes as special assumption. I myself have always supposed that only statements, and hypotheses insofar as they lead through deductions to statements, could contradict one another. The view that facts and events could themselves be in contradiction seems to me to be a prime example of careless thinking. The foregoing remarks are intended only to establish the fact that the definitive clarification of the nature of the infinite, instead of pertaining just to the sphere of specialized scientific interests, is needed for the dignity of the human intellect itself. From time immemorial, the infinite has stirred men's emotions more than any other question. Hardly any other idea has stimulated the mind so fruitfully. Yet, no other concept needs clarification more than it does.

...

The theory of proof which we have here sketched not only is capable of providing a solid basis for the foundations of mathematics but also, I believe, supplies a general method for treatment fundamental mathematical questions which mathematicians heretofore have been unable to handle.

In a sense, mathematics has become a court of arbitration, a supreme tribunal to decide fundamental questions on a concrete basis on which everyone can agree and where every statement can be controlled. The assertions of the new so-called "intuitionism" modest though they may be must in my opinion first receive their certificate of validity from this tribunal. An example of the kind of fundamental questions which can be so handled is the thesis that every mathematical problem is solvable. We are all convinced that it really is so. In fact one of the principal attractions of tackling a mathematical problem is that we always hear this cry within us: There is the problem, find the answer; you can find it just by thinking, for there is no ignorabimus in mathematics. Now my theory of proof cannot supply a general method for solving every mathematical problem there just is no such method. Still the proof (that the assumption that every mathematical problem is solvable is a consistent assumption) falls completely within the scope of our theory. I will now play my last trump. The acid test of a new theory is its ability to solve problems which, though known for a long time, the theory was not expressly designed to solve. The maxim "By their fruits ye shall know them" applies also to theories. When Cantor discovered his first transfinite numbers, the so-called numbers of the second number class, the question immediately arose, as I already mentioned, whether this transfinite method of counting enables one to count sets known from elsewhere which are not countable in the ordinary sense. The points of an interval figured prominently as such a set. This question whether the points of an interval, i.e., the real numbers, can be counted by means of the numbers of the table given previously is the famous continuum problem which Cantor posed but failed to solve. Though some mathematicians have thought that they could dispose of this problem by denying its existence, the following remarks show how wrong they were: The continuum problem is set off from other problems by its uniqueness and inner beauty. Further, it offers the advantage over other problems of combining these two qualities: on the one hand, new methods are required for its solution since the old methods fail to solve it; on the other hand, its solution itself is of the greatest importance because of the results to be obtained. The theory which I have developed provides a solution of the continuum problem. The proof that every mathematical problem is solvable constitutes the first and most important step toward its solution…

[At this point, Hilbert sketched an attempted solution of the continuum problem. The attempt was, although not devoid of interest, never carried out. We omit it here. Eds.]

In summary, let us return to our main theme and draw some conclusions from all our thinking about the infinite. Our principal result is that the infinite is nowhere to be found in reality. It neither exists in nature nor provides a legitimate basis for rational thought a remarkable harmony between being and thought. In contrast to the earlier efforts of Frege and Dedekind, we are convinced that certain intuitive concepts and insights are necessary conditions of scientific knowledge, and logic alone is not sufficient. Operating with the infinite can be made certain only by the finitary. The role that remains for the infinite to play is solely that of an idea if one means by an idea, in Kant's terminology, a concept of reason which transcends all experience and which completes the concrete as a totality that of an idea which we may unhesitatingly trust within the framework erected by our theory.

3. Freges **Anti-Psychologism**

I'm taking this from the entry **Psychologism** *in the Stanford Encyclopedia of Philosophy: [3]. That entry begins:*

> Many authors use the term psychologism for what they perceive as the mistake of identifying non-psychological with psychological entities. For instance, philosophers who think that logical laws are not psychological laws would view it as psychologism to identify the two. Other authors use the term in a neutral descriptive or even in a positive sense. Psychologism then refers (approvingly) to positions that apply psychological techniques to traditional philosophical problems.

It is worth reading all of that entry. But I've copied below the following part of it for our discussion; this is section 4 of [3]:

Consider first Frege's *Grundlagen der Arithmetik* (1884). One of Frege's main thesis is that mathematics and logic are not part of psychology, and that the objects and laws of mathematics and psychology are not defined, illuminated, proven true, or explained by psychological observations and results. One of Frege's central arguments for this thesis is the consideration that whereas mathematics is the most exact of all sciences, psychology is imprecise and vague. Thus it is implausible to assume that mathematics could possibly be based upon, or be a part of, psychology. A closely related further point is that we need to distinguish between psychological ideas (Vorstellungen) and their objects. This distinction is especially important when the latter are objective or ideal. Numbers, for example, are objective and ideal entities, and thus they differ fundamentally from ideas. Ideas are always subjective and idiosyncratic. In this context Frege laments the fact that the term idea has also been used for objective, essentially non-sensual, abstract and objective entities . Frege rejects psychological or physiological interpretations of the Kantian distinctions between the a priori and the a posteriori, and the analytic and the synthetic; as Frege has, it these distinctions concern different ways in which judgments are justified or proven true, not different operations of the human mind.

- **J.S. Mill:**
  A central theme of the Grundlagen is a detailed criticism of Mill's philosophy of mathematics. Frege argues that mathematical truths are not empirical truths and that numbers cannot be properties of aggregates of objects. First, Frege denies Mill's claim that mathematical statements are about matters of fact. Frege's objection is that there is no physical fact of the matter to which the numbers 0 or 777864 refer. Moreover, someone who learns how to calculate does not thereby gain any new empirical knowledge. Second, Frege insists that there is no general inductive law from which all mathematical sentences can be said to follow. Third, while Frege grants Mill that some empirical knowledge may well be necessary for us to learn mathematics, he points out that empirical knowledge cannot justify mathematical truths. Fourth, Frege counters Mill's claim

according to which numbers are properties of aggregates of objects with the observations that aggregates do not have in and of themselves characteristic manners in which they can be divided. Frege also points out that the numbers 0 and 1 are not aggregates at all. And finally, Frege accuses Mill of overlooking that numbers can be predicated of both concrete and abstract objects. Frege continues his criticism of psychological logic in the Foreword of his *Grundgesetze der Arithmetik* (1893). He begins by pointing out that the word law is ambiguous: it can refer to descriptive or to prescriptive laws. Examples of the former are the laws of physics, examples of the latter are moral laws. Frege suggests that every descriptive law can be reformulated as a prescription to think in accordance with it. For instance, true descriptive laws ought to be accepted. Hence they can be prefixed by the prescription: 'accept the truth of . . . .' In other cases the prescription might be a set of instructions on how to reach the truth stated in a descriptive law. Frege's main point in all this is that whereas all prescriptive laws can be categorized as 'laws of thought' on the basis that they tell what we ought to think, only one kind of descriptive laws deserves to be come under the same heading: the set of psychological descriptive laws. Frege claims that in the realm of logic we find both descriptive and prescriptive laws, with the former being the foundation for the latter. The point merits stressing since it is sometimes suggested that for Frege the opposition between psychological laws and logical laws is the is-ought opposition. But note that Frege writes: every law that states what is can be apprehended as prescribing that one ought to think in accordance with it This holds of geometrical and physical laws no less than logical laws. Thus logical laws are primarily descriptive laws even though, like other descriptive laws, they too can be reformulated or apprehended as prescriptive laws. These distinctions allow Frege to distinguish between two versions of psychological logicians: one group takes the laws of logic to be descriptive psychological laws, the other group interprets the laws of logic as prescriptive laws based on descriptive psychological laws. It is the ambiguity of the expression law of thought that invites these confusions. Frege's main criticism of psychological logic is that it conflates true and being-taken-to-be-true.

To begin with, Frege denies that prescriptions based on psychological laws can qualify as proper logical laws. Such prescriptions can be no more than demands to conform to current thinking habits. But they cannot be yardsticks by which these thinking habits can be evaluated as to their truth. Moreover, Frege points out that the descriptive psychological laws which for the psychological logician provide the basis for (psycho-)logical prescriptions are laws of taking-to-be-true: they state the conditions under which humans accept the truth of judgments or the validity of inferences; but they do not determine the conditions under which judgments are true and inferences valid.

- **B. Erdmann**

  Here Frege is particularly scathing of the psychological logician Benno Erdmann who identifies truth with general consensus. For Frege this move makes truth dependent upon what-is-taken-to-be-true. And it fails to give proper heed to the insight that truth is independent of people's agreement. It follows that the laws of logic are not psychological laws: 'If being true is thus independent of being acknowledged by somebody or other, then the laws of truth are not psychological laws: they are boundary stones set in an eternal foundation, which our thought can overflow but never displace'. Frege's attack on Erdmann does not end here. Erdmann is also taken to task for his suggestion that logical laws might have mere hypothetical necessity, that is, that they are relative to the human species. Frege maintains that if we were to encounter creatures who deny the laws of logic, we would take them to be insane; and he analyses Erdmann's proposal as reducing, yet again, truth to what-is-taken-to-be-true. At the same time, Frege accepts that the most fundamental logical laws cannot be justified. Logical justification comes to an end when we reach these laws. However, to argue that our nature or constitution forces us to abide by the laws of logic is no longer a logical justification; it is to incorrectly shift from logical to psychological or biological considerations. For Frege the opposition between truth and what-is-taking-to-be-true is closely linked to the distinction between accepting and rejecting the realm of objective and non-real entities. Frege insists that the realm of the non-real is not identical with the realm of the psychological and subjective. His example of objective, non-psychological entities are numbers. Numbers are not ideas since they are the same for all subjects. Moreover, Frege tries to convince us that the denial of the objectivity and non-reality of numbers and concepts leads fairly directly into idealism and solipsism. Since psychological logicians try so hard to break down the distinction between the realms of the objective-ideal and the subjective-psychological, they easily are tempted to go further and challenge the borderline between the subjective-psychological and the objective-real. The resulting standpoint is idealism and solipsism: idealism since the only existing entities are ideas; solipsism since all ideas are relative to subjects. And thus the possibility of communication is a further victim of their efforts. Again Frege is eager to show that Erdmann's logic is guilty of the charges. He therefore points out that Erdmann calls both hallucinated objects and numbers objects of an ideal nature; that Erdmann fails to distinguish between acts and contents of judgments; and that Erdmann lacks the conceptual resources to distinguish between ideas and realities. Frege's alternative is of course to emphasize that coming-to-know is an activity that grasps rather than creates objects. This choice of terminology is meant to bring out that what we come to know is (usually) independent of us. After all, when we grasp a physical object like a pencil, the object is independent both of the act of grasping and of the human actor of the grasping.

- **Edmund Husserl**

  Frege's third anti-psychologistic discussion is his 1894-review of Husserl's Philosophie der Arithmetik (1891). I shall not here enter into the debate over whether it was Frege's criticism that turned Husserl away from psychologism. Nor shall I try to adjudicate whether Frege's critique was justified. Suffice it here to point out that Frege's review classifies Husserl as a psychological logician on the grounds that Husserl treats the meanings of words, concepts and objects as different kinds of ideas; that Husserl provides psychological-genetic accounts of the origins of abstract concepts ; and that Husserl, like Erdmann, equivocates on the notion of idea: in some places in his book, concepts and objects are understood as subjective, in other places they are taken to be objective. In his criticism of Husserl's psychological explanation of the genesis of the number concept, Frege does not confine himself to contrasting Husserl's theory with his own account of numbers. He also points out that the various psychological processes which Husserl's theory assumes are spurious. For instance, Frege denies that we can combine any arbitrary contents into one idea without relating these contents to one another. He also rejects Husserl's claim that we can abstract from all differences between two contents and still retain their numerical distinctness (1894, 316, 323).

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5 Notes for Oct 21 session: Objectivity and Subjectivity in Mathematics

In Mathematics, the issue (objective versus subjective) spans attitudes labelled *mathematical platonism, intuitionism, formalism.*

- Mathematical platonism views mathematical substance as having an essence independent of human thought; as being part of a pre-noetic structure of the cosmos; and the aim of mathematics is to faithfully describe it.
- Intuitionism, in its various forms, puts the spotlight on the manner in which mathematics is actually thought.[17]
- (The Kantian take on this is marvelously subtle: not easily classifiable.)

[17] There is quite a range of literature about this. Here's one interesting example: [3].

- A formalist approach focusses specifically on the *language* of mathematics as holding the key to its meaning.

Given that our time is short, and I would like to make room for discussion, I'll reserve a few important aspects of our topic for some later discussion. Each of these, if gone into appropriately, would take an entire session:

- The issue: *What is a set?*
- The mathematician **Kurt Gödel** who was a key 'player' in the drama we are about to discuss, and was a supreme mathematical platonist. (He will make only a brief appearance below.)
- The general question of *mathematical induction* (This also only briefly appears below.)

See also [6], [7], [8] for different takes on this material.

5.1 What is the number *'A thousand and one'*?

On the one hand, it is an object of thought—e.g., of my thought. John Locke might call it an "*object* of understanding" (following his discussion in Chapter XXI *Of the Division of the Sciences* in his *Essay Concerning Human Understanding*). This Locke frames as being in contrast to thoughts about

> "things in [my] own power, which are [my] own actions, for the attainment of [my] own ends."

On the other hand, if asked what that number actually *is*, one would be obliged to come up with some sort of definition.

Perhaps I'd say "it is one more than a thousand" which would, in some sense be a tautological reshuffling of the vocabulary of the original question, involving concepts (*one, a thousand, more than*) themselves begging for definition.

But—at the very least—such a response would be suggestive of a recursive ploy where I might be defining things in terms of other things—*a thousand* is *one more than nine hundred ninety-nine—and so on,* invoking turtles all the way down… finally relating it all to an undefined primitive (e.g.: *the unit*, whatever that is) via a process deemed legitimate and comprehensible.

There is an inescapable tinge of *subjectivity* to all this.

5.2 Plato

Plato is usually touted as the standard-bearer of objectivity—e.g., viewing mathematics as something like a study of the architecture of the cosmos, where "out there" is the emoji of a phrase that captures this feeling: the objects of mathematics are *out there* described—but not created—by human thought.

Nevertheless, even Plato—in *The Republic* VI.510c,d—has Socrates offering Adeimantus a vivid sense of the quite subjective practice of mathematicians[18]:

> *… the men who work in geometry, calculation, and the like treat as known the odd and the even, the figures, three forms of angles, and other things akin to these in each kind of inquiry.*
>
> *These things they make* **hypotheses** *and don't think it worthwhile to give any further account of them to themselves or others as though they were clear to all. Beginning from them, they … make the arguments for the sake of the square itself and the diagonal itself, not for the sake of the diagonal they draw, and likewise with the rest. These things themselves that they mold and draw—shadows and images in water—they now use as images, seeking to see those things themselves, that one can see in no other way than with thought.*

[18] as opposed, admittedly, to the dialectic practices of philosophers

5.3 'Tower Pound' definition

Perhaps I could fashion a definition of the number *A thousand and one* in some grossly material way following the definition of the *Tower Pound* whose *standard* (a pound of gold, I think) was kept in the Royal Mint in the Tower of London. I might secure in a closet a certain collection of what I proclaim to be *a thousand and one objects*—and use that collection as a *standard*. Any collection of objects deserves to be called *a thousand and one objects* if (and only if) this collection can be put in one-one correspondence with the standard collection (in my closet).

5.4 J.S. Mill

This attitude toward number finds resonance in, for example, the writings of John Stuart Mill[19]:

> All numbers must be numbers of something: there are no such things as numbers in the abstract (System, VII: 254).

And, in any event, all reasoning is fundamentally discursive:

> Since Reasoning, or Inference, the principal subject of logic, is an operation which usually takes place by means of words, and in complicated cases can take place in no other way; those who have not a thorough insight into the signification and purposes of words, will be under chances, amounting almost to certainty, of reasoning or inferring incorrectly

[19] *A System of Logic, Rationative and Inductive, being a Connected View of the Principles of Evidence and the Methods of Scientific Investigation* Harper & Brothers 1882

5.5 Georg Cantor:

The concept of *one-one correspondence* employed in the 'Tower Pound" version of the meaning of that number (as given in Subsection 5.2 above) must lurk—I think—behind any discussion of the meaning of number. It surely is at the heart of Georg Cantor's definition of *cardinality*[20]:

A **one-one correspondence** between two sets S and T is given by a mapping

$$f : S \longrightarrow T \tag{1}$$

that is *injective* (i.e., no two different elements of S map to the same element in T under the mapping f) and *surjective* (every element of T is the image of some element of S).

Equivalently, the mapping f in Equation (1) is a **one-one correspondence** if it has an inverse; i.e., if there exists a mapping in the reverse direction $g : T \longrightarrow S$ such that the compositions $fg : T \to T$ and $gf : S \to S$ are the identity mappings.

Two sets are defined to be of the same **cardinality** if (and only if) there is a one-one correspondence between them.

The shocker, in Cantor's Theory[21], is the existence of many *infinite* cardinalities, the basic example being demonstrated by what is known as his **diagonal proof**. Namely, the proof of his theorem asserting that the set of (positive, say) real numbers less than 1 (e.g., the set whose elements are infinite decimals decimals $0.a_1a_2a_3\ldots a_n\ldots$ representing real numbers in that range) can never be counted—that is, this set cannot be put in one-one correspondence with the set of whole numbers $1, 2, 3, \ldots$.

More precisely, if you give any proposed listing of such real numbers $r_1, r_2, r_3, \ldots,$ Cantor—by his "Diagonal Proof,"—will give you a specific real number r that

[20] and this he formulates in a setting far less 'material' than 'Tower Pounds'

[21] His work spans two decades: from the late 1870s to the late 1890s.

is not on your list.[22].

5.6 Gottlob Frege ($\sim$ 1900)

would certainly disapprove of any "Tower Pound" definition of the number '*a thousand and one*' for—surely—many reasons, the least of which is its dependence on 'material objects.'

1. Quoting from the readings for today's session ([3]):

> A central theme of Frege's *Grundlagen* is a detailed criticism of Mill's philosophy of mathematics. Frege argues that mathematical truths are not empirical truths and that numbers cannot be properties of aggregates of objects.
>
> - First, Frege denies Mill's claim that mathematical statements are about matters of fact. Frege's objection is that there is no physical fact of the matter to which the numbers 0 or 777864 refer. Moreover, someone who learns how to calculate does not thereby gain any new empirical knowledge.
> - Second, Frege insists that there is no general inductive law from which all mathematical sentences can be said to follow.
> - Third, while Frege grants Mill that some empirical knowledge may well be necessary for us to learn mathematics, he points out that empirical knowledge cannot justify mathematical truths.
> - Fourth, Frege counters Mill's claim according to which numbers are properties of aggregates of objects with the observations that aggregates do not have in and of themselves characteristic manners in which they can be divided. Frege also points out that the numbers 0 and 1 are not aggregates at all.

[22] And even more precisely, Cantor's *Diagonal Proof* is enacted by the following somewhat comical scenario: For you to"give" your listing, you might do this, presumably, in some organized temporal fashion, such as producing the first m digits of each of the first m numbers you are listing, $r_1, r_2, \ldots, r_m$, say, by the m-th day; and do this for $m = 1, 2, \ldots$ and so on. Well Cantor' strategy is to produce on the m-th day the first m digits of *his* number r just by choosing the i-th digit of his r to be any digit different from the i-th digit of r_i, for all $\imath = 1, 2, \ldots, m$. This is a winning strategy for him, since his r will clearly never be one of your r_i's for any i.

- And finally, Frege accuses Mill of overlooking that numbers can be predicated of both concrete and abstract objects.

2. But another reason for Frege to object to my 'Tower Pound definition' is its ad hoc-ness—it reeks of personal choice, of subjective whim; if you wish, of *psychologism*. Frege preferred to remove anything ad hoc—anything of involving some personal choice—from the definition of number (e.g., definition of *a thousand and one*)—or from his definition of any concept.

   The format of his *Grundlagen* would give him the freedom[23] to defined 'cardinality' (alias: a 'number' which might possibly be infinite) to simply be *the equivalence class* of **all sets of the same cardinality**. For example, the number **2** *is defined to be*—Frege wanted to say—the set of all couples; the number **3** *is* the set of all triples.

   The problem—as was already something that Cantor worried about, and as was dramatically pointed out by Bertrand Russell[24]—is that there are paradoxes that arise if you allow yourself to define sets by formulating a property $p(x)$ and then by stipulating:

   > the set of all objects x having some specific property, $p(x)$.

   I.e., by *unrestricted universal quantification.*

   The safe thing to do is to restrict universal quantification to the objects of some previously defined set:

   > the set of all objects x in the set Ω having some specific property, $p(x)$.

   Bertrand Russell brought this home by his famous paradox—the too curious set $\mathcal{X}$ defined as:

$$\mathcal{X} := \ \{\text{sets } x \mid x \notin x\}.$$

[23] He includes an axiom in his system of logic that, for any property $p(x)$ allows him to form the set of all x having that property.

[24] For a few neat slides giving some of the historical interchange between Frege and Russell, see [1].

5.7 Foundations... or *Constitutions*

In an article for a popular audience [4] the mathematician Michael Harris suggested, in passing, that one might view the grounding of mathematics as dependent upon 'foundations' (which, of course, is the usual view) or perhaps upon a *constitution*. The question of which choice of term one focuses on—*foundation* or *constitution*—is rather topical for our seminar: pitting the presumed objectivity of foundations of a discipline against the subjectivity involved in setting up a constitution: therefore dealing with the hurly-burly of some founding constitutional convention, an event involving personality clashes perhaps that require bargains to be made.

5.8 David Hilbert

It is pretty clear on which side of this divide Hilbert positions himself. This one gets from almost any sentence of his essay *On The Infinite* [2]; e.g.:

> The goal of my theory is to establish once and for all the certitude of mathematical methods.

Hilbert, in that essay, is—in effect—defining a new mathematical object—**formal systems**—destined to play a double role in mathematics:

- for any mathematical theory there should be an appropriate formal system to serve as a framework for it; allowing one to have confidence in the deductions that can be made by the rules of that formal system,
- formal systems themselves can be taken to be mathematical objects in their own right and can be studied to shed light on the nature of rigorous deduction.

Some decades later, John Von Neumann, writing to Rudolph Carnap, proclaims:

> [Kurt] Gödel has shown the unrealizability of Hilbert's program. ... There is no more reason to reject intuitionism (if one disregards the aesthetic issue, which in practice will also for me be the decisive factor).

And von Neumann had previously written to Gödel:

> I think that your result has solved negatively the foundational question: there is no rigorous justification for classical mathematics.

Returning to Hilbert's text, he is arguing that the core requirement of *formal systems* is just that they be consistent—meaning that you can't get contradictions using the rules of procdure of that formal system: you can't prove both a proposition P and its negation $\neg P$. Once you are assured of this, the formal system is serviceable.

Having sketched this notion, Hilbert continues:

> **The theory of proof which we have here sketched not only is capable of providing a solid basis for the foundations of mathematics but also, I believe, supplies a general method for treating fundamental mathematical questions which mathematicians heretofore have been unable to handle.** In a sense, mathematics has become a court of arbitration, a supreme tribunal to decide fundamental questions on a concrete basis on which everyone can agree and where every statement can be controlled. The assertions of the new so-called "intuitionism" modest though they may be must in my opinion first receive their certificate of validity from this tribunal.
>
> ...
>
> An example of the kind of fundamental questions which can be so handled is the thesis that every mathematical problem is solvable.

> We are all convinced that it really is so. In fact one of the principal attractions of tackling a mathematical problem is that we always hear this cry within us: There is the problem, find the answer; you can find it just by thinking, for **there is no ignorabimus in mathematics.** Now my theory of proof cannot supply a general method for solving every mathematical problem there just is no such method. Still the proof (that the assumption that every mathematical problem is solvable is a consistent assumption) falls completely within the scope of our theory.

This is precisely what Gödel showed is *not the case.*

It may have been David Hilbert who actually introduced the phrase *axiomatic thinking* to signal the fundamental role that the structure of an axiomatic system plays in mathematics. Hilbert clearly views himself as molding a somewhat new architecture of mathematical organization in his 1918 article "Axiomatisches Denken." It begins with a political metaphor, that neighboring sciences being like neighboring nations need excellent internal order, but also good relations one with another, and:

> ... The essence of these relations and the ground of their fertility will be explained, I believe, if I sketch to you that general method of inquiry which appears to grow more and more significant in modern mathematics; the axiomatic method, I mean.

Hilbert's 1918 essay ends:

> In conclusion, I should like to summarize my general understanding of the axiomatic method in a few lines. I believe: **Everything that can be the object of scientific thinking in general, as soon as it is ripe to be formulated as a theory, runs into the axiomatic method and thereby indirectly to mathematics.**

> Forging ahead towards the ever deeper layers of axioms in the above sense we attain ever deepening insights into the essence of scientific thinking itself, and we become ever more clearly conscious of the unity of our knowledge. In the evidence of the axiomatic method, it seems, mathematics is summoned to play a leading role in science in general.

5.9 L.E.J. Brouwer

Going in quite a different direction is Brouwer's *intuitionist* take on foundations. Hilbert's *On The Infinite* was actually an attempt to shore up Cantor's set theory agains the attack launched by Brouwer:

> Aus dem Paradies, das Cantor uns geschaffen, soll uns niemand vertreiben können. (From the paradise, that Cantor created for us, no-one shall be able to expel us.)[25]

Here is an excerpt of a Wikipedia entry in *Intuitionism*:

> The fundamental distinguishing characteristic of intuitionism is its interpretation of what it means for a mathematical statement to be true. In Brouwer's original intuitionism, the truth of a mathematical statement is a subjective claim: a mathematical statement corresponds to a mental construction, and a mathematician can assert the truth of a statement only by verifying the validity of that construction by intuition. The vagueness of the intuitionistic notion of truth often leads to misinterpretations about its meaning.

The Wikipedia entry goes on to say:

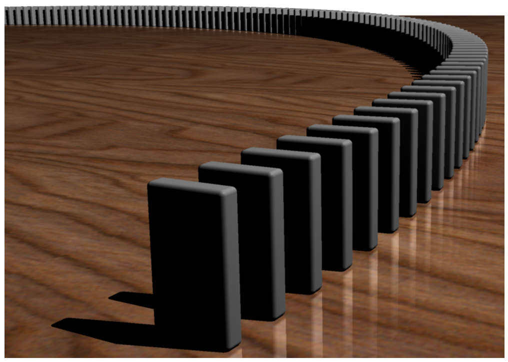
> Intuitionistic truth therefore remains somewhat ill-defined.

[25] This is from a lecture Hilbert gave in Münster to the Mathematical Society of Westphalia in 1925.

I would prefer to say that it simply has a *subjective element* to it. But the entry roughly captures, I think, the essence of a general intuitionist's position:

> However, because the intuitionistic notion of truth is more restrictive than that of classical mathematics, the intuitionist must reject some assumptions of classical logic to ensure that everything they prove is in fact intuitionistically true. This gives rise to intuitionistic logic.
>
> To an intuitionist, the claim that an object with certain properties exists is a claim that an object with those properties can be constructed.**Any mathematical object is considered to be a product of a construction of a mind, and therefore, the existence of an object is equivalent to the possibility of its construction. This contrasts with the classical approach, which states that the existence of an entity can be proved by refuting its non-existence.** For the intuitionist, this is not valid; the refutation of the non-existence does not mean that it is possible to find a construction for the putative object, as is required in order to assert its existence. As such, intuitionism is a variety of mathematical constructivism; but it is not the only kind.
>
> The interpretation of negation is different in intuitionist logic than in classical logic. In classical logic, the negation of a statement asserts that the statement is false; to an intuitionist, it means the statement is refutable (e.g., that there is a counterexample). There is thus an asymmetry between a positive and negative statement in intuitionism. If a statement P is provable, then it is certainly impossible to prove that there is no proof of P.

5.10 The simple phrase "*and so on...*"



When I defined *a thousand* at the beginning of Section 5.1 above as *one more than nine hundred ninety-nine—and so on,* I was invoking a bit of *mathematical induction*. Of course, it is a "downward induction" and the implied inductive process would end finitely—even though to say this already has a suspicion of circularity ingrained in it since, after all, we're in the midst of defining that finite number.

Mathematical induction is a key concept in the various narratives here and, as one might expect, there is a wide range of attitudes towards it.

The most extreme position towards induction is offered by the *utrafinitists* who would even not be all that happy with the number *a thousand and one* that we talked about at the beginning of today's session. **Yessinin-Volpin** is one such. He was, for many reasons, an extraordinarily interesting human being . See his Wikipedia page [9]. In it there is an account of a conversation Volpin had with the mathematical logician Harvey Friedman. Friedman asked Volpin if he 'believed in' the numbers $2^1, 2^2, 2^3, \dots 2^{100}, \dots$. Friedman writes:

> He asked me to be more specific. I then proceeded to start with 2^1 and asked him whether this is "real" or something to that effect. He virtually immediately said yes. Then I asked about 2^2, and he again said yes, but with a perceptible delay. Then 2^3, and yes, but with more delay. This continued for a couple of more times, till it was obvious how he was handling this objection. Sure, he was prepared

to always answer yes, but he was going to take 2^{100} times as long to answer yes to 2^{100} then he would to answering 2^1. There is no way that I could get very far with this.

References

[1] http://brianrabern.net/onewebmedia/Frege-Russell.pdf

[2] Hilbert, D. On the Infinite https://math.dartmouth.edu/~matc/Readers/HowManyAngels/Philosophy/Philosophy.html

[3] Psychologism (Stanford Encyclopedia of Philosophy) https://plato.stanford.edu/entries/psychologism/

[4] Harris, M., Why the Proof of Fermat's Last Theorem Doesnt Need to Be Enhanced https://www.quantamagazine.org/why-the-proof-of-fermats-last-theorem-doesnt-need-to-be-enhanced-2019

[5] Mathematical Platonism and its Opposites https://canvas.harvard.edu/courses/59645/files?preview=8541387

[6] Terence Tao, "What is Good Mathematics?", Bulletin of American Mathematical Society, 44 (4), 623-634, 2007.

[7] Timothy Gowers, "The Two Cultures of Mathematics", Mathematics: Frontiers and Perspectives, Edited by V. I. Arnold, Michael Atiyah, Peter D. Lax and Barry Mazur, American Mathematical Society, 2000.

[8] William Thurston, "On Proof and Progress in Mathematics", Bulletin of American Mathematical Society, 30 (2), 161-177, 1994.

[9] Yessinin-Volpin https://en.wikipedia.org/wiki/Alexander_Esenin-Volpin

6 Readings for October 30, 2019: *(Shades of Objectivity and Subjectivity in Epistemology, Probability, and Physics)*

1. **Analytic versus Synthetic; A priori versus A posteriori**

   Professor Sen asked me to include in my October 30 session some discussion of the Kantian notions that are the title of this section. These ideas have had enormous influence. I feel they are priceless primers for discussion—a discussion that I hope we will have in our session. I also feel that they constitute a beautiful cathedral, somewhat in ruin. Wonderful to roam around. I'm not sure that it is wise to restore or renovate—as people have tried to do. But important to think about. The Wikipedia entry:

   `https://en.wikipedia.org/wiki/Analytic%E2%80%93synthetic_distinction#Quine's_criticisms`

   is very good. Read especially the first two sections.

2. **Subjective Probability**

   Read the handout: *'Educating your beliefs' versus 'Testing your Hypotheses'*

3. **A 'subjective view' of Physics**

   Read pages 1-9 of Mach's *Science of Mechanics*[26]. In these few pages, Mach expresses his quite extraordinary view of the nature of physical laws, and of the way in which experience and economy-of- thought play their role in our understanding and formulating them. Also take a look at pages 10-20 where he begins to show how he intends to use these views to understand physical laws.

[26] A link giving the 1974 paperback—Open Court Publishing Co.— edition of this work will be on the class web-site.

7 Notes for October 30, 2019: *(Shades of Objectivity and Subjectivity in Epistemology, Probability, and Physics)*

1. *Analytic* versus *Synthetic*[27]; *a priori* versus *a posteriori*

   To begin our discussion, consider the 'statements' given in the list below. Imagine that each of them was actually made by some person (it's cleaner if you think of these as people who you don't know at all).

   The question is: judging from the nature of the statements themselves, what types of *preparations* (e.g., in thought or activity) and what types of *resources* (mental, or physical) do you think would be necessary for a person to come up with each of these statements?

   (a) It's hotter today than yesterday.
   (b) If you let go of a ball it will fall.
   (c) A triangle is a polygon having three sides.
   (d) 5 plus 1 is 6.
   (e) All bachelors are unmarried.
   (f) If the proposition P implies the proposition Q, and if P is true, then Q is true.
   (g) "All bodies are heavy"
   (h) A triangle is a polygon having three angles.
   (i) The sum of the angles of a triangle is 180 degrees.
   (j) 5 plus 2 equals 7.
   (k) The only consecutive numbers that are perfect powers (i.e. squares, cubes, or higher powers of numbers) are the numbers 8 and $8+1 = 9$.

   Kant aimed to fit statements of the above sorts into distinct categories

[27] The entry *The Analytic/Synthetic Distinction* in the Stanford Encyclopedia of Philosophy: `https://plato.stanford.edu/entries/analytic-synthetic/` is very much worth reading.

of thought[28], categories that depend on the answers to the question we posed above. The reason for such an exercise is not so much to merely provide a "Linnaean type clasification" of statements, but rether to get a more vivid sense of the nature of the mental faculties that are required, to make such statements with conviction.

Clearly (a), (b) requires some more pointed engagement with the world than does the rest of the statements. Merriam-Webster's definitions of **a posteriori** offer a reasonable description of this type of engagement:

> **(1)** Relating to or derived by reasoning from observed facts,
> **(2)** 'Inductive.'

There is a huge difference between the statements (a) and (b). They correspond to the two definitions—**(1)** or **(2)**—offered by Merriam-Webster: Statement (a) is a straightforward empirical assessment, established—perhaps—by having checked a thermometer yesterday, and doing it again today. Statement (b), though, calls up full scientific induction; i.e., is a prediction based on repeated similar experiences with the expectation that they continue to have similar outcomes—this type of assessment (to channel Hume) being one of our *habits of thought*.

And either of these entries—**(1)** or **(2)**—are in accord with Kant's meaning of the descriptive term *a posteriori*: judgments that are *validated by, and grounded in, experience.*

Statements (c) and (d) are of a different nature, (c) being a straight definition, and (d)—but depending, a bit, on your experience with numbers—is also a definition. Neither statements require any appeal to empirical observations, or data. They are *a priori* statements—following Google's (and Kant's) definition of *a priori*:

> relating to or denoting reasoning or knowledge which proceeds

[28]The above list doesn't cover the full range of Kantian 'categories of types of statements.' E.g.: *This dahlia is beautiful.* is a different sort of thing from anything on the list.

from theoretical deduction rather than from observation or experience.

Insofar as (c) and (d) are definitions, the concepts on either side of the word "is" (in each of these statements) are simply the same. These are 'analytic (a priori) propositions':

Definition 1. *An* **analytic proposition** *is a proposition whose predicate concept is contained in its subject concept.*

Statement (e) is an *analytic a priori* statement as well: it requires no appeal to empirical observations, and it conforms to Definition 1 even though not all unmarried people are bachelors. The label *analytic* here simply acknowledges that no mental faculty on a par with those that Kant calls *intuitions* are necessary to see that the *predicate concept* is—in fact—contained in its *subject concept*.

Statement (f) is visibly 'analytic" and 'a priori.'

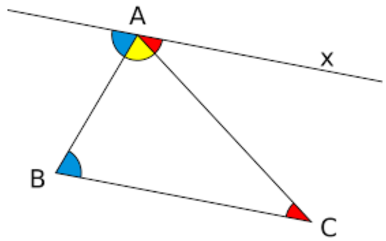
Statement (g)—dependent on empirical justification is visibly 'a posteriori'. But the predicate subject (*heavy*) is not 'contained,' in any obvious way, in the subject concept (*analytic*) and Kant calls it *synthetic*[29].

Definition 2. *A* **synthetic proposition** *is a proposition whose predicate concept is not contained in its subject concept but is related to the subject concept (by means of some empirical justification, and/or making use ofthe mental faculties that Kant labels: space, time, and causality).*

As for (h): it is surely 'a priori': no empiricism is necessary: no reference to anything other than the concepts in that statement is required. But, taking (c) as the *definition* of triangle, a tiny bit of shuffling—in thought—around the sides of a triangle Δ is in fact necessary to see that Δ has three angles. Would that engage our intuition of space (and perhaps time) sufficiently to get Kant to categorize Statement (g) as *synthetic*? I think so.

[29]in *Critique of Pure Reason* A7/B1

Statement (i)—equally visibly 'a priori'— is even more evidently dependent on some mental resource; namely the idea of performing a construction such as in the diagram:



I want to leave Statements (j) and (k) for discussion.

2. In the Preface to the Second Edition of the *Critique of Pure Reason*[30] Kant describes his 'Copernican Revolution' carving out a niche for his *synthetic a priori*:

   The examples of mathematics and natural science, which by one revolution have become what they now are, seem [xvi] to me sufficiently remarkable to induce us to consider, what may have been the essential element in that intellectual revolution which has proved so beneficial to them, and to make the experiment (at least, so far as the analogy between them, as sciences of reason, with metaphysics allows it) of imitating them. Hitherto it has been supposed that all our knowledge must conform to the objects: but, under that supposition, all attempts to establish anything about them a priori, by means of concepts, and thus to enlarge our knowledge, have come to nothing. The experiment therefore ought to be made, whether we should not succeed better with the problems of metaphysics, by assuming that the objects must conform to our mode of cognition, for this would better agree with the demanded possibility of an a priori knowledge of them, which is to settle something about objects, before they are given us.

   We have here the same case as with the first thought of Copernicus, who, not being able to get on in the explanation of the movements of the heavenly bodies, as long as he assumed that all the stars turned round the spectator, tried, whether he could not succeed better, by assuming

[30] https://oll.libertyfund.org/titles/ller-critique-of-pure-reason

the spectator to be turning round, and the stars to be at rest. A similar experiment may be tried in metaphysics, so far as the intuition of objects is concerned.

If the intuition had to conform to the constitution of objects, I do not see how we could know anything of it a priori; but *if the object (as an object of the senses) conforms to the constitution of our faculty of intuition, I can very well conceive such a possibility.* . . . I must. . . , even before objects are given to me, presuppose the rules of the understanding as existing within me a priori, these rules being expressed in concepts a priori, to which all objects of experience must necessarily conform, and with which they must agree. With regard to objects, [xviii] so far as they are conceived by reason only, and conceived as necessary, and which can never be given in experience, at least in that form in which they are conceived by reason, we shall find that the attempts at conceiving them. . . will furnish afterwards an excellent test of our new method of thought, according to which we do not know of things anything a priori except what we ourselves put into them.

8 Subjectivity and Objectivity in Statistics: 'Educating your beliefs' versus 'Testing your Hypotheses'

The naive view of an empirical investigation which we might call the **straight Baconian model** for a scientific investigation has, as recipe:

Set-up and Hypotheses $\longrightarrow$ **Data Collecting** $\longrightarrow$ **Processing Data and Conclusion.**

The manner in which one proceeds from data to conclusion is often understood to be a straight comparison of what the hypotheses would predict and what the data reveals[31], the comparison being (usually) quantitative with a pre-specified tolerance of discrepancy (between prediction and observation).

[31] although it might be difficult to find this expressed in Bacon's writings as bluntly

All this is significantly modified by the Bayesian viewpoint, which methodically intertwines the first two steps, and has a different take on each of these ingredients: hypothesis, data, conclusions. We'll discuss this below[32]. We'll look at the Bayesian viewpoint as offering a 'model' to help us understand, and deal with, the interplay between those ingredients. Let's call it the **Bayesian model** for a scientific investigation.

A further issue that complicates the contrast of *models of getting to scientific conclusions* alluded to above is the difference between the Bayesian's and the Frequentist's work; their methods are not the same, and they have slightly different *primary goals.* The Bayesian starting point is to offer tentative probability distributions that one expects describes the Data (accumulated so far, and continuing to be accumulated). Such a *tentative probability distribution* is meant merely to start the procedure—a best initial guess—and (appropriately) called a "prior." The grand function of the continuing accumulation of data is for this data to be "fed back to educate the prior"—changing it perhaps— but retaining it as a probability distribution (which we'll call a "posterior.") The movement here is as follows:

$$\text{Prior (probabilities)} \quad \xrightarrow{\textbf{Data}} \quad \text{Posterior (probabilities).}$$

This, of course, is going to form a loop, where as data gets accumulated, the prior gets 'educated,' and rendered therefore (one hopes) a better indication of phenomena.

The **black box**—so far—is that I have not yet said anything about the mathematical procedure Bayesians use to feed back (as an afterburner) information obtained by the Data into the prior assumptions, in order to effect the "education" of these prior assumptions and thereby produce the posterior. For the moment—in this discussion—it is more important for me simply

[32]A disclaimer: I know very little statistics; I'm a total outsider to this field and especially to the extended conversation—and the somewhat sharp disagreements—that Bayesians and Frequentists have. Whatever is in this section of my notes I learned from the statistician Susan Holmes—all the errors, though, are of my own creation.

to emphasize that *whatever this procedure is* it is, in fact, a *predetermined procedure.*

8.1 Predesignation versus the self-corrective nature of inductive reasoning

Now you might well worry that this Bayesian ploy is like curve-fitting various hypotheses[33] to the data—a kind of hypothesis-fishing expedition, if you want. You keep changing the entire format of the problem, based on accumulating data. The Bayesians have, as I understand it, a claim: that any two 'reasonable' priors, when "corrected" by enough data will give very close posteriors. That is, the initial rough-hewn nature of the prior will iron out with enough data. Their motto:

> *Enough data swamps the prior.*

I've been playing around with another formulation of that motto:

> *Any data-set is, in fact, a 'data point' giving us information about the probability distribution of priors.*

In contrast, there is a motto that captures the sentiment of a Frequentist:

> *Fix hypotheses. This determines a probability distribution to be expected in the data. Compute data. If your hypotheses are good,* **in the limit** *the data should conform to that probability distribution.*

About the above, one of the early great theorizers in this subject (and specifically regarding probability, randomness, and the law of large numbers) was

[33] I want to use the word *hypothesis* loosely, for the moment; that is, the way we generally use the word; and not in the specific manner that statisticians use it.

Jacob Bernoulli. He *also* was a theologian preaching a specifically Swiss version of Calvinism. You see the problem here! There is a strict vein of *predetermined* destiny or fatalism in his theology, someone who is the father of the theory of randomness. How does he reconcile these two opposites? Elegantly, is the answer! He concludes[34] his treatise *Ars Conjectandi*, commenting on his law of large numbers, this way:

> Whence at last this remarkable result is seen to follow, that if the observations of all events were continued for the whole of eternity (with the probability finally transformed into perfect certainty) then everything in the world would be observed to happen in fixed ratios and with a constant law of alternation. Thus in even the most accidental and fortuitous we would be bound to acknowledge a certain quasi necessity and, so to speak, fatality. I do not know whether or not Plato already wished to assert this result in his dogma of the universal return of things to their former positions [apokatastasis], in which he predicted that after the unrolling of innumerable centuries everything would return to its original state.

Apokatastasis is a theological term, referring to a return to a state before the fall (of Adam and Eve)[35].

Also, we might connect the above with C.S. Peirce's 1883 paper "A Theory of Probable Inference"[36]. Peirce makes a distinction between *statistical deduction* and *statistical induction* the first being thought of as reasoning from an entire population to a sample, and the second being reasoning from sample to population. In the first it is a matter of long run frequency (i.e, the Frequentist's motto) whereas the second is related to a Peircean conception

[34] It is, in fact, the conclusion of the *posthumously* published treatise (1713) but it isn't clear to me whether or not he had meant to keep working on the manuscript.

[35] Noah Feldman once suggested to me that Calvinists might be perfectly at home with random processes leading to firm limiting fatalism, in that the fates of souls—in Calvinist dogma—are *randomly assigned* and not according to any of their virtues; i.e., to misquote someone else: "goodness had nothing to do with it."

[36] For a readable discussion of this paper, see: Len O'Neill's *Peirce and the Nature of Evidence* published in the Transaction of the Charles S. Peirce Society **29** Indiana Univ. Press (1993) pp. 211-224.

of *the self-corrective nature of inductive reasoning* (and this sounds like the Bayesian protocol).

Peirce dwells on the issue of *predesignation* in the Frequentist's context (i.e., you fix a model and then collect evidence for or against it; you don't start changing the model midstream in view of the incoming evidence). As already mentioned, there is a curious type of *meta-predesignation* in the Bayesian context, in that the manner in which you change the model, given incoming evidence, is indeed pre-designated.

Extending this, one might think of any (pre-designated) recursive format that provides successive approximations to a sought-for limit as something of an allegory of the Bayesian viewpoint.

8.2 Priors as 'Meta-probabilities'

Suppose you are a cancer specialist studying a specific kind of cancer and want to know if there is a gender difference: do more men than women get this type of cancer? Or more women than men?

Now suppose I asked you (cancer specialist) to make some kind of guess—when considering groups of people that get this cancer—about the proportion of men-to-women that get it. You might tabulate this as a probability P that a random choice of person in this group is male. So P is a number between 0 and 1. You might actually give me a number if you are very confident, but more likely, for a spread of possible values of P, you'll give me an estimate of greater or lesser levels of confidence you have that this P is indeed the sought-for-probability. Taking the question I asked more systematically, you might interpret it as follows:

> As P ranges through all of its possible values, from 0 (no males get it) to 1 (only males get it) tell me (your guess of) the probability that P is the ratio $\frac{M}{M+W}$ where M is the number of men and W the number of women in the group? In effect, draw me a graph telling

your probability-estimate for each of the P's in the range between 0 and 1.

Your initial guess, and initial graph, is the Prior (I privately call it the *meta-probability*). It *will* be educated by the data accumulating.

Let's imagine that you say "I have no idea! This probability P could–as far as I know–equally likely be any number between 0 and 1." If so, and if you had to draw a graph illustrating this noncommittal view, you'd draw the graph of a horizontal line over the interval $[0,1]$. Or, you might have some reason to believe that P is close to $1/2$ but no really firm reason to believe this and you might have no idea whether gender differences enter at all. Then the graph describing your sense of the likelihood of the values of P would be humped symmetrically about $P = 1/2$. Or if you are essentially certain that it is $1/2$ you might draw it to be symmetrically spiked at $P = 1/2$.

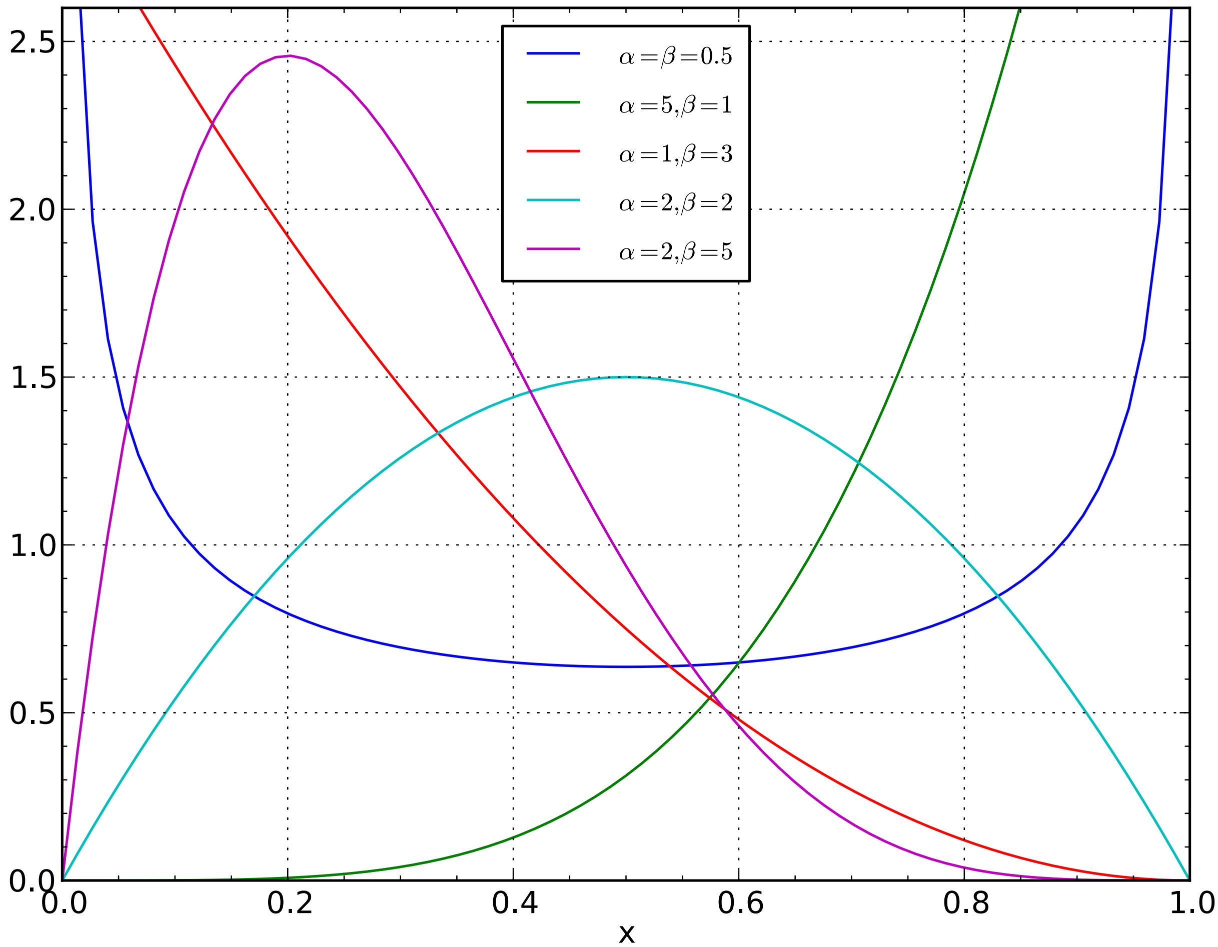
What you are drawing is–in a sense–a *meta-probability density* since you are giving a portrait of your sense of how probable you think each value between 0 and 1 might be the actual probability-that men-get-this-type-of-cancer. Your portrait is the graph of some probability density function $f(t)$.

There are theoretical reasons to suggest, for some such problems, that you would do well to be drawing the graphs of a specific well-known family called **beta-distributions**. These beta-distributions come as a two parameter family[37] $\beta_{a,b}(t)$. That is, fix any two positive numbers a, b (these numbers a, b are called the *shape parameters* of the beta-distribution) and you get such a graph.

Here are some general ground-rules for choosing these βs: shape parameters that are equal give distributions symmetric about $1/2$; i.e., you choose such a β if you expect that gender plays no role in the probability of contracting this cancer. Choosing $a > b$ means that you are skewing things to the left; i.e., you believe that men get this type of cancer less frequently than women; choosing $b > a$ means the reverse. The larger these parameters, the sharper the peak of the curve; i.e., the more "sure" you are that the probability occurs at the peak.

[37] These are distributions $t^{a-1}(1-t)^{b-1}dt$ normalized to have integral equal to 1 over the unit interval.

Choose parameters, say, $a = 2, b = 5$; or, say, $a = 2, b = 2$ and you have probability distributions $\beta_{2,5}(t)$, or $\beta_{2,2}(t)$, these being the blue and the magenta graphs in the figure below.



8.3 Back to our three steps

1. (**Choosing the Prior**) Now, Bayesian cancer doctor that you are, when you start doing your statistics, choose a Prior. For this type of question you might do well, as I said, to choose some beta-distribution. If you imagine that there might be a gender bias here, but have no idea in

which direction, you might choose one that is symmetric about $t = 1/2$ (which, as it turns out, means that you'd be taking shape parameters a equal to b). But size up the situation as best as you can, taking into account everything that you think is important to the problem and come up with a choice of a Prior. Let us say that your Prior is $\beta_{a,b}(t)$.

2. (**The Data**) Suppose you now get a data sample of 100 people with cancer—perhaps the result of some specific study of some particular population, and suppose that 60 of these cancer victims are men (so 40 are women).

3. (**Passing to the Posterior**) The beauty of the family of beta-distributions is that when you appropriately *educate* a beta-distribution (the Prior) with new data, the new distribution (the Posterior) is again a beta-distribution. The only thing is that the shape parameters may change; say, from (a, b) to a new pair of numbers (a', b'):

$$\beta_{a,b}(t) \quad \xrightarrow{\text{new data}} \quad \beta_{a',b'}(t)$$

   I'm told that this change can be very easily computed. That is, in this example problem, the a', b' will depend on hardly more than the original a, b, the percentage of men with cancer, and the size of the study.

8.4 A numerical example and a question

For this example I'm normalizing things so the numbers work simply so we don't get bogged down in mere arithmetic. Imagine that your Prior is $\beta_{20,20}$ and you test a sample population (of just the right size for the normalizations to work out as I'm going to assume they do below) and in that population Men/ Women cancer ratio is 60/40. The Posterior is then (I'm told) $\beta_{20+60,20+40}$. And if you compute (based on that Posterior) the probability that men get this type of cancer more than women, that probability is:

$$0.955\ldots$$

If you did the analogous thing with the Prior $\beta_{10,10}$, getting, as Posterior, $\beta_{10+60,10+40}$ you'd compute (based on that Posterior) the probability that men get this type of cancer more than women to be:

$$0.966\ldots$$

Question: Why is it *reasonable* that the second estimate of probability of gender-difference be bigger than the first?

9 Issues of Subjectivity and Objectivity in Physics

Ernst Mach's view of Physics may be a good start for a good discussion about this. Here is an excerpt from Chapter 1 of his *Science of Mechanics.*

> When we wish to bring to the knowledge of a person any phenomena or processes of nature, we have the choice of two methods : we may allow the person to observe matters for himself, when instruction comes to an end; or, we may describe to him the phenomena in some way, so as to save him the trouble of to save him the trouble of personally making anew each experiment. Description, however, is only possible of events that constantly recur, or of events that are made up of component parts that constantly recur. That only can be described, and conceptually represented which is uniform and conformable to law; for description presupposes the, employment of names by which to designate its elements; and names can acquire meanings only when applied to elements that constantly reappear.

In the infinite variety of nature many ordinary events occur; while others appear uncommon, perplexing, astonishing, or even contradictory to the ordinary run of things. As long as this is the case we do not possess a well-settled and unitary conception of nature. Thence is imposed the task of everywhere seeking out in the natural phenomena those elements that are the same, and that amid all multiplicity are ever present. By this means, on the one hand, the most economical and briefest description and communication are rendered possible ; and on the other, when once a person has acquired the skill of recognising these permanent elements throughout the greatest range and variety of phenomena, of seeing them in the same, this ability leads to a *comprehensive, compact, consistent* and *facile conception of the facts*. When once we have reached the point where we are everywhere able to detect the same few simple elements, combining in the ordinary manner, then they appear to us as things that are familiar; we are no longer surprised, there is nothing new or strange to us in the phenomena, we feel at home with them, they no longer perplex us, they are explained. It is a process of adaptation of thoughts to facts with which we are here concerned.

Economy of communication is of the very essence of science. Herein lies its pacificatory, its enlightening, its refining element. Herein, too, we possess an unerring guide to the historical origin of science. In the beginning, all economy had in immediate view the satisfaction simply of bodily wants. With the artisan, and still more so with the investigator, the concisest and simplest possible knowledge of a given province of natural phenomena a knowledge that is attained with the least intellectual expenditure naturally becomes in itself an economical aim; but though it was at first a means to an end, when the mental motives connected therewith are once developed and demand their satisfaction, all thought

To find, then, what remains unaltered in the phenomena ot nature, to discover the elements thereof and the mode of their interconnection and interdependence this is the business of physical science. It endeavors, by comprehensive and thorough description, to make the waiting for new experiences unnecessary; it seeks to save us the trouble of experimentation, by making use, for example, of the known interdependence of phenomena, according to which, if one kind of event occurs, we may be sure beforehand that a certain other event will occur. Even in the description itself labor may be saved, by discovering methods of describing the greatest possible number of different objects at once and in the concisest manner. All this will be made clearer by the examination of points of detail than can be done by a general discussion.

10 Consequentialism of Meaning—notes for part of session of December 4

In today's session, Professor Sen will be discussing the issue of *intention* versus *consequence*. I.e., the broad implications of *consequentialism*, its affect on our deliberations regarding how we choose to act, and the relevant history of discussion about this. Without taking too much time from the session—but in an attempt to piggy-back on the subject—I would like to frame a striking moment in the history of mathematics[38] as being captured by the phrase

a consequentialism of meaning.

That moment occurred in 1925 when Hilbert delivered his address *On The Infinite* before a congress of the Westphalian Mathematical Society in Munster, in honor of Karl Weierstrass. This address was 'suggested reading' for an earlier session, but it might be fun to review it again, focusing on it now as an act of 'mathematical consequentialism.'

Hilbert offered a (then: *amazing, novel*) idea that is nowadays greeted with an *of course!*

[38] Or perhaps I should say the *history of the philosophy of mathematics*

To compare with what might be taken to be universally assumed, before Hilbert, consider René Descartes' (unfinished) treatise—begun in 1628—*Regulae ad directionem ingenii* or *Rules for the Direction of the Mind*, where the general procedure appropriate for rational argument in science (as promoted by Descartes in a dozen meticulously formulated 'rules') has a step-by-step format, each step justified, and completely understood, before the next step is taken:

$$\boxed{\text{Hypotheses}} \rightarrow \boxed{\text{Step 1}} \rightarrow \boxed{\text{Step 2}} \rightarrow \ldots \rightarrow \boxed{\text{Step n}} = \boxed{\text{Conclusion}}$$

Here's Descartes' *Rule VIII*:

> **Rule VIII:** If in the series of things to be examined we come across something which our intellect is unable to intuit sufficiently well, we must stop at that point, and refrain from the superfluous task of examining the remaining items.

In contrast, Hilbert is insistent that intermediate steps in mathematical demonstrations be super-precise, but he doesn't require them to mean *anything* beyond their formal notation—they are mere symbolic entities that don't have to carry any meaning—as long as they lead to "correct" conclusions.

The scare-quotes around "correct" signify that even that word carries a novel interpretation: *correct* means *consistent*; i.e., *doesn't lead to a contradiction.* In sum, it is a consequentialist take on the very notion of 'meaning.' That is, the *means* by which one gets to the conclusion need not be anything more than making correct moves in a symbolic language—i.e., following ordinary, but explicitly ordained and rigorously formulated, logical operations—and the only thing that need be taken into consideration is:

Does this explicitly procedure lead to a contradiction or not (if so, it is useless; if not, it is a guide to the truth):

$\boxed{\text{Hypotheses}} \to \boxed{\text{precise procedure within formal system}} \to \boxed{\text{Conclusion}}$

Formal systems provide a format for algorithmic procedures:[39]

$\boxed{\text{Input}} \to \boxed{\text{explicit algorithm}} \to \boxed{\text{Output}}$

Here is Hilbert:

> The goal of my theory is to establish once and for all the certitude of mathematical methods. This is a task which was not accomplished even during the critical period of the infinitesimal calculus. This theory should thus complete what Weierstrass hoped to achieve by his foundation for analysis and toward the accomplishment of which he has taken a necessary and important step.... But a still more general perspective is relevant for clarifying the concept of the infinite. A careful reader will find that the literature of mathematics is glutted with inanities and absurdities which have had their source in the infinite. **For example, we find writers insisting, as though it were a restrictive condition, that in rigorous mathematics only a finite number of deductions are admissible in a proof as if someone had succeeded in making an infinite number of them.** Also old objections which we supposed long abandoned still reappear in different forms. For example, the following recently appeared: **Although it may be possible to introduce a concept without risk, i.e., without getting contradictions, and even though one can prove that its introduction causes no contradictions to arise, still the introduction of the concept is not thereby justified.** Is not this exactly the same objection which was once brought against complex-imaginary numbers when it was said: "True, their use doesn't lead to contradictions. Nevertheless their introduction is unwarranted, for imaginary magnitudes do not exist"? **If, apart from proving consistency, the question of the justification of a measure is to have any mean-**

[39] and some 'machine-learned' algorithms present interesting case-studies regarding the manner in which the intermediate steps of the process are relatively inaccessible to our understanding

ing, it can consist only in ascertaining whether the measure is accompanied by commensurate success. Such success is in fact essential, for in mathematics as elsewhere success is the supreme court to whose decisions everyone submits. **As some people see ghosts, another writer seems to see contradictions even where no statements whatsoever have been made, viz., in the concrete world of sensation, the "consistent functioning" of which he takes as special assumption. I myself have always supposed that only statements, and hypotheses insofar as they lead through deductions to statements, could contradict one another. The view that facts and events could themselves be in contradiction seems to me to be a prime example of careless thinking.** The foregoing remarks are intended only to establish the fact that the definitive clarification of the nature of the infinite, instead of pertaining just to the sphere of specialized scientific interests, is needed for the dignity of the human intellect itself. From time immemorial, the infinite has stirred men's emotions more than any other question. Hardly any other idea has stimulated the mind so fruitfully. Yet, no other concept needs clarification more than it does.

The theory of proof which we have here sketched not only is capable of providing a solid basis for the foundations of mathematics but also, I believe, supplies a general method for treating fundamental mathematical questions which mathematicians heretofore have been unable to handle. In a sense, mathematics has become a court of arbitration, a supreme tribunal to decide fundamental questions on a concrete basis on which everyone can agree and where every statement can be controlled. The assertions of the new so-called "intuitionism" modest though they may be must in my opinion first receive their certificate of validity from this tribunal.

...

> An example of the kind of fundamental questions which can be so handled is the thesis that every mathematical problem is solvable. We are all convinced that it really is so. In fact one of the principal attractions of tackling a mathematical problem is that we always hear this cry within us: There is the problem, find the answer; you can find it just by thinking, for **there is no ignorabimus in mathematics.** Now my theory of proof cannot supply a general method for solving every mathematical problem there just is no such method. Still the proof (that the assumption that every mathematical problem is solvable is a consistent assumption) falls completely within the scope of our theory.

This is precisely what Gödel showed is *not the case.*

11 Dealing with nonexistent objects

Suppose that you are perfectly happy with positive numbers, but refuse—as some people in prior centuries refused—to 'believe' (whatever that means) that **negative numbers** (whatever that means) 'exist' (whatever that means)?

Can you do essentially everything that you might do, if you were content to use work-arounds, such as ledger books where what other people would have negative numbers you would treat the corresponding positive numbers as 'debits,' so to speak and happily deal with them as such? Periphrasis! I suppose this is possible; i.e., that there is a certain pliability in how we need to proceed, in terms of assigning the label **exist** to certain candidate-concepts.

Going in a different direction, mathematicians in the midst of some process of showing that a certain mathematical object doesn't exist *(I'm hesitant to call such a 'thing' a certain mathematical object...)*—well, mathematicians assume they exist, and happily try to discover various properties that they possess, piling up those properties in the hope that it will determine, in the end, that the 'thing' doesn't exist. Some of these 'things' actually have pet names (*the 'ghost zero' of certain L-functions being one famous example*).

I hope we can have a fruitful discussion about this curious issue—nonexistent objects being conceived and being studied, exhibiting various properties—without having any need to know explicitly such a technical example.

There seems to be an allowance here for a somewhat subjective aspect to the pinning of the label *exist* to concepts. We mentioned earlier in our course the hefty tradition of discussion regarding the existence (or at least the definition) of God. These arguments can be essentially pro- (i.e., claiming that God's existence is proved) as in St. Anselm, or Spinoza; or essentially critical as in Aquinas or Kant. An enlightening account of these arguments, and their history, can be found in the Stanford Encyclopedia of Philosophy's entry `https://plato.stanford.edu/entries/ontological-arguments/`. (Read especially sections 1-3.)

A curious common thread in many of the ontological arguments is to allow 'existence' to be a possible predicate (or not!) of the various things-of-thought. St. Anselm, for example, puts a value judgment on this predicate :*it's more perfect to exist than not!*[40]. Compare this with Spinoza's Definition IV quoted in the citation above.

So if you conjure the most perfect thing-of-thought that can be conceived, well: if it doesn't exist, there's your contradiction. For now imagine whatever it is that you conjured up, but as *existing*, and you've just conceived of a yet more perfect thing-of-thought—voilá.

Often in these ontological arguments one sees the *unqualified* use of the quantifier $\exists$ to establish *existence* (of something) as being a predicate (of that something). That is: one asserts existence of an entity, without specifying in what realm that entity is (so-to-speak) 'taken from.'

In symbols: as long as you have a set in mind as your domain of discourse—call it Ω—it makes sense to consider formulas such as:

$$\exists x \in \Omega \ \text{ such that } \ \ldots,$$

but you're asking for trouble if you have no specific set such as Ω in mind

[40] *Old joke:*
A: I wish I never was born!
B: Oh, only one in a million is that lucky.

and just want to deal with the formula:

$$\exists x \text{ such that } \ldots .$$

(This puts such arguments in the same framework as *unqualified* use of the quantifier $\forall$, as is behind Russell's paradox and the various uses of 'unqualified universal quantification, related to the classical *crisis in the foundations of mathematics.*)

Going back to the literature regarding *God's existence*, Baruch de Spinoza, however, in his *Ethics* has—as far as I can make out—a unique take. Spinoza gives three different 'proofs'.

1. Two versions of the 'ontological argument':
   - God's essence (simply) entails existence.
   - The potentiality of non-existence is a negation of power, and contrariwise the potentiality of existence is a power, as is obvious.

2. A version of the principle of insufficient reason:

   > If, then, no cause or reason can be given, which prevents the existence of God, or which destroys his existence, we must certainly conclude that he necessarily does exist. If such a reason or cause should be given, it must either be drawn from the very nature of God, or be external to him—that is, drawn from another substance of another nature. For if it were of the same nature, God, by that very fact, would be admitted to exist. But substance of another nature could have nothing in common with God, and therefore would be unable either to cause or to destroy his existence.

A curious argument ...