

On Numbers and the World

Barry Mazur

A review of *Numbers and the World (Essays on Math and Beyond)*
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Phillip Lopate writes¹ “The great promise of essays is the freedom they offer to explore, digress, [and] acknowledge uncertainty . . .” In this book, Mumford, one of my oldest friends, has selected some entries from his blog in which he muses thoughtfully about a wide range of topics, motivated primarily by the unfolding of his curiosity. He has a gift for putting two or more disparate thoughts and structures together in a manner that both connects them and elucidates them.

Like the essayist Montaigne whose *Book the first* has chapters on intention, sorrow, idleness, and liars, Mumford’s selection of topics range widely – the book is broken into six parts: Opening More Eyes to Mathematics; The History of Mathematics; AI, Neuroscience, and Consciousness; And Now, Some Bits of Real Math; Coming to Terms with the Quantum; and Nothing is Simple in the Real World.

Among the chapters of *Numbers and the World* we find the Pythagorean Theorem, Grothendieck’s Theory of Schemes, the literacy and numeracy of Babylonian goddesses, modern art and modern math, rogue waves, DNA, Spinoza, robots, and much more. To give a sense of the book, I will focus on a few of the themes treated in Mumford’s essays. It is a book better simply to read than be told about.

1 Beauty and Mathematics

You do mathematics because you want to apply your findings to real life questions or you are seeking some pure understanding. The ‘surprise’ is often the emergence of a structure that is exciting, that is beautiful. This has an echo—except for the forbidding role played by *duty*—in the attitude expressed by the Prioress in Isak Dinesen’s *Seven Gothic Tales*²:

Straight is the line of duty,

Curved is the line of beauty.

Follow the straight line; thou shalt see

The curved line ever follows thee.

Mumford contemplates the question *what is a beautiful mathematical formula?* He recalls that at one point two different collaborative teams of mathematicians had independently sought to answer this question – but in two radically different ways. The first – described in the website www.concinnitasproject.org – is by producing and exhibiting art objects that exemplify the striking beauty of mathematical formulae. The second team’s aim was to map via fMRI scans the brain activity of people as they think about a piece of mathematics to establish correlations with the inner experience of beauty that they say they experience.

¹<https://myfivethings.com/class/reading-and-writing-the-essay/>

²p. 116 in *Seven Gothic Tales*, Modern Library, Random House, 1934.

Mumford interweaves an account of this with personal reflections about his family and their close connection with Art. He also considers how the grand emergence of Abstract Art and the flourishing of Abstract Mathematics occur at the same time. This might be a reflection of a more suffuse ‘flavor of the time,’ a *Zeitgeist*.

He writes:

The discovery that randomness can be harnessed to create both math and art seems to have taken place in the short period 1945-1950. (p. 65)

Jackson Pollock was at the forefront in engaging with the beauty of randomness. His paint hitting the canvas captures at that instant the rush of beauty, of the same sort of vibrancy shared with ocean waves as they clash against a rocky shore – or, for that matter, as the hydrodynamical equations that attempt to describe this type of activity.

Randomness is a major theme in the book. Mumford offers the thought (in Chapter 13. An Applied Mathematician’s Foundations of Math, and elsewhere in his writings) “that random variables must be treated as a third type of variable along with real numbers and integers.” (p. 153)

What might that mean? Well, a number is just a . . . number. A random variable, if you want to think of it as a number, is a *number* that will be the result of some action, but nevertheless can be an item of discussion even before that action is completed. In other words, its value, if we can use the word value, is determined by some probability distribution. For example, it would be the exact position that your dart will hit the dartboard, able to be thought about even before you throw the dart.

In an illuminating footnote (p. 154), Mumford likens the idea of ‘taking random variables as numbers’ as being analogous to generic points in algebraic geometry. Let’s take an example. Any complex number z in the complex plane gives rise to a prime ideal in the ring $\mathbb{C}[X]$ of all polynomials in the variable X with complex number coefficients, namely, the ideal generated by the linear polynomial $X - z$. This is, in fact, a maximal ideal in that ring. Taking the quotient of the ring $\mathbb{C}[X]$ by the ideal generated by $X - z$ *imposes the condition that $X = z$* , that is, it specializes the variable X to the complex number z . All maximal ideals in this ring are of that form, so we can simply say: the complex plane consists of nothing more than the set of the maximal ideals in that ring, the quotient by each ideal *specializing* the variable X to a specific complex number.

But this discussion leaves out one prime ideal in $\mathbb{C}[X]$, namely the ideal $\{0\}$, the ideal generated by the ‘trivial polynomial,’ which is indeed a prime ideal. It is contained, of course, in all prime ideals. Passing to the quotient by this ideal $\{0\}$, does absolutely nothing, has no “specializing effect” on the variable X . It is natural, then, to dub $\{0\}$ the *generic point* of the complex plane. We can think of it as hovering over every honest point in the complex plane, in that it, as an ideal, is contained in all the other ideals.

There is, of course, a more general framework. For any commutative ring R consider the set of *all* of its prime ideals. Call that set $\text{Spec}(R)$; any element of this set can be thought of as a *point*. Let’s say that any one of those points *specializes to*, that is, it hovers over another point Q if the prime ideal P is contained in the prime ideal Q . (Hence passing to the quotient by the ideal Q is a stricter focus of specialization than passing to the quotient by the ideal P .) In this general context there can be quite an intricate constellation of points hovering over other points. The idea, then, that Mumford considers, of ‘taking random variables as numbers’ might be to imagine a more fluid ‘Spec structure’ where random variables are given a status analogous to numbers, and they are thought to hover, according to a probability distribution, over naive numbers.

Richard Dedekind in his 1858 essay *Continuity and Irrational Numbers* introduced what we now call ‘Dedekind cuts,’ referring to that construction as the “creation of new *point-individuals*.” He wrote that the real line is infinitely richer in *point-individuals* than the set of rational numbers in

number-individuals. Mumford’s suggestion creates an infinitely rich assortment of ‘*random-variable-individuals*.’

2 What kinds of mathematicians are there?

The traditional categorization of mathematicians is pretty simple: it ranges from *Formalists* who are happy in broad axiomatic settings, allowing for nonconstructive – and nonconstructed – mathematical concepts convoked by the axiom of choice to *Finitists* like Yessinin-Volpin [6] who might hesitate for a second before conceding that 2^{100} exists.

Mumford has his own more vivid categorization of mathematical thinkers. He views them in *tribes* depending on “what most strongly drives them in their esoteric world” (p. 27) and he labels these tribes *explorers*, *alchemists*, *wrestlers* and *detectives*.

Mumford thinks of himself as a member of a sub-clan of explorers that he calls *mappers*.

According to Mumford, *mappers* are mathematicians who are keen on surveying the whole range of possibilities of a species of mathematical objects. Mappers make ‘maps’ of all examples of the particular type of mathematical object they are interested in understanding, the idea being that if you take them all together, they may – and they often do – reveal a deeper structure than they do when taken one by one. Such *maps* are called *moduli spaces*.

Mumford is the foremost artisan of the fundamental moduli spaces that form the core of modern algebraic geometry. His essential constructions, for example, of the moduli spaces of projective curves of genus g , or of principally polarized abelian varieties, are prime examples of the beauty of the subject, and powerful tools to help us understand algebraic varieties.

To give a sense of what is involved in forming such a moduli space, consider this simple mathematical concept taught in High School geometry classes: *similar plane triangles*. Two triangles in the plane are called **similar** if after taking one of them and zooming its size, translating, rotating, and flipping it in the plane appropriately, you can bring it to fit precisely onto the other one.

Can we find a ‘map’ that has the property that each point on the map corresponds to a similarity class of plane triangle, and each similarity class corresponds to a point on the map? Moreover will it be true, in an appropriate sense, that the very structure of the map *respects* the relationship that similarity classes have to one another?

The answer to that question is yes: take a plane triangle $\Delta \subset \mathbf{R}^2$ as a representative of its class of triangles that are similar to it. By appropriate ‘zooming’ we can replace Δ so that its longest edge is of length 1. This hasn’t changed its similarity class. Now move Δ by translating and rotating appropriately so that its edge of length 1 is given by the unit interval on the x -axis,³ and flip the plane by an appropriate flip, for example, $(x, y) \mapsto (x, \pm y)$, or $(x, y) \mapsto (1 - x, \pm y)$, so that the image of Δ now has its ‘third vertex’ $v = (x, y)$ lying in $M :=$ the piece of the unit disc defined by:

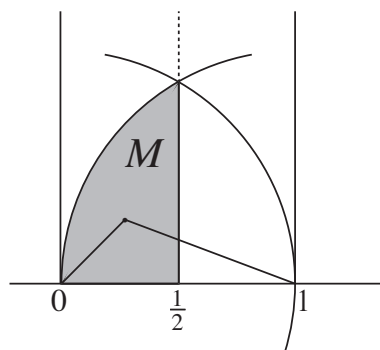
$$\{(x, y) \in \mathbf{R}^2 \mid 0 \leq x \leq \frac{1}{2}, \ y > 0 \text{ and } (x - 1)^2 + y^2 \leq 1\}.$$

None of these maneuvers have changed the similarity class of our Δ .

This ‘piece of disc’ M constitutes a *map of all similarity classes of plane triangles*. For a point $v \in M$ build a triangle Δ_v with vertices $(0, 0)$, $(0, 1)$ and v . As v runs through all the points of our map M , the triangles Δ_v will run through representatives of every similarity class, and if $v \neq v'$ then Δ_v and $\Delta_{v'}$ will be dissimilar—that is, will represent different similarity classes. Moreover, if two points on our map are close, then the similarity classes they represent are also, in an appropriate

³and if it is isosceles or equilateral, then do this for one of its edges of length 1

Figure 1: The moduli space of equivalence classes of similar plane triangles



sense, close. Note that similarity classes of isosceles triangle correspond to the boundary points of our map M and the class of the equilateral triangle has the honor of being the topmost point.

Of course, this is an extremely simple example, merely a hint of what it means to have a map (a ‘moduli space’) that parametrizes all instances of some mathematical species.

Quite often it happens in practice that the moduli space of a species of mathematical object has an even richer structure than the species it is meant to explain. And understanding the inner structure of moduli spaces may have great rewards; for example, a thorough comprehension of a certain relevant moduli space led to the proof, by Andrew Wiles and Richard Taylor, of Fermat’s Last Theorem.

3 Getting acquainted with the prime numbers, a few at a time

Mumford’s Chapter 11, *Finding the Rhythms of the Primes*, is in the part of the book that begins with the warning: “And now for some real math” and, indeed, more mathematical background is required to follow it than is required in other parts of the book. Mumford considers one of the fundamental achievements of Pure Mathematics: a theory now over a century and a half old yet remains rife with open problems, the most well-known open problem being something called the *Riemann Hypothesis*. I won’t explicitly enter this vast mathematical territory, except to say that the prime numbers

$$2, 3, 5, 7, 11, 13, 17, \dots$$

are the *atoms* in the world of *Numbers*; any whole number greater than 1 is a molecule built uniquely as a product of atoms. How the primes are distributed in the sequence

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \dots$$

of all numbers is one of the great mathematical questions. The number of prime numbers less than or equal to any number X – usually called $\pi(X)$ – can be (surprisingly, and somehow) explicitly given by a formula expressed in terms of another infinite set of (rather strange) complex numbers, namely, the nontrivial zeros of the Riemann zeta-function. The Riemann Hypothesis states that these zeroes are of the form $\frac{1}{2} + \omega\sqrt{-1}$ where ω is a real number.⁴

⁴This hypothesis appears in Riemann’s 1859 paper [5]—but there it is not labelled ‘hypothesis’ or ‘conjecture’ but it is clearly Riemann’s expectation:

“... it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some futile attempts, as it appears unnecessary for the next objective of my investigation.”

Since $\frac{1}{2} + \omega\sqrt{-1}$ is a zero of the Riemann zeta-function if and only if $\frac{1}{2} - \omega\sqrt{-1}$ is a zero, we can organize the information given by these roots $\frac{1}{2} \pm \omega\sqrt{-1}$ simply by considering the increasing sequence of positive real numbers, $\omega_1, \omega_2, \omega_3, \dots$, that appear as “ ω ”s.

These real numbers are colloquially referred to as the ‘spectrum’ of the prime numbers. Any of these numbers is easily computed to any order of accuracy one wishes:

$$\begin{aligned}\omega_1 &= 14.134725\dots \\ \omega_2 &= 21.022040\dots \\ \omega_3 &= 25.010858\dots \\ \omega_4 &= 30.424876\dots \\ \omega_5 &= 32.935062\dots \\ \omega_6 &= 37.586178\dots\end{aligned}$$

The reason for the label *spectrum* is that an appropriately normalized (infinite) trigonometric sum involving $\cos(\omega_i x)$ for $i = 1, 2, 3, \dots$ converges⁵ to a function of the real variable x that has discontinuities exactly when x is a power of a prime number—that is, a function that reveals, by its very basic structure, the placement of prime numbers in the real number line. The explicit connection between the *arithmetic structure* that is the placement of the set of prime numbers on the real line and the *analytic structure* that is its spectrum is given by a relation called the von Mangoldt Formula,

$$\boxed{\{2, 3, 5, 7, 11, \dots\} \quad \longleftarrow \text{von Mangoldt Formula} \quad \longrightarrow \quad \{\omega_1, \omega_2, \omega_3, \dots\}}$$

The utility of this formula can’t be overestimated: by performing a certain amount of analysis related to the *spectrum*, the right-hand-side of that formula, one produces analytic expressions that approximate the growth of $\pi(X)$, the essence of the left-hand-side of that formula. Moreover, those analytic approximating expressions get closer and closer to the function $\pi(X)$.

Mumford goes in the opposite direction. Utterly enjoying the experimentation in the manner, he says, of an applied mathematician he uses just a few prime numbers to produce approximations of the first few ω s. For example he gets 14.185... as an approximation to $\omega_1 = 14.1347\dots$. He invites the reader to play with this piece of mathematics, writing that

... the basic idea behind Riemann’s periodic terms is indeed apparent in these small primes. This is especially startling because the convergence of the explicit formula is very slow: there are very many rapidly oscillating terms beyond the first one so there is no compelling reason why the lowest ω_k should nail these primes this well. This suggests there might be other formulas relating the primes with the zeros clarifying this correspondence.(p. 128)⁶

4 Independence of Scale.

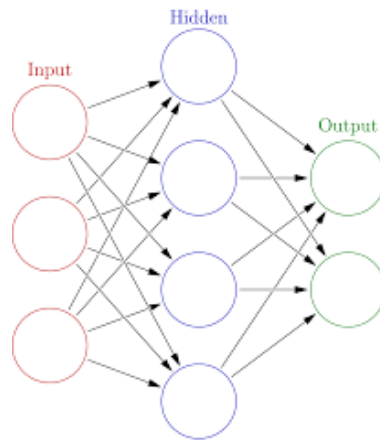
D’Arcy Thompson, in his book *On Growth and Form*⁷ quotes Galileo commenting that *scale matters* ([4] p. 19):

⁵In a manner that needs specific definition.

⁶In this same direction, see the recently published paper *Aperiodic diffraction grating based on the relationship between primes and zeros of the Riemann zeta function* by A.E. Madison, D.A. Kozodaev, A.N. Kazankov, P.A. Madison, V.A. Moshnikov, Technical Physics **69** No. 4 (2024) 620-624.

⁷Published by Cambridge University Press (1917).

Figure 2: A Neural Net



... it was Galileo who, well nigh 300 years ago, had first laid down this general principle which we now know by the name of the principle of similitude; and he did so with the utmost possible clearness, and with a great wealth of illustration, drawn from structures living and dead. He showed that neither can man build a house nor can nature construct an animal beyond a certain size, while retaining the same proportions and employing the same materials as sufficed in the case of a smaller structure. The thing will fall to pieces of its own weight unless we either change its relative proportions, which will at length cause it to become clumsy, monstrous and inefficient, or else we must find a new material, harder and stronger than was used before.

Mumford notes that mouse and human brains are strikingly similar in structure (and, to some small extent, also in function), even though they are strikingly different in size. The mouse has about 13 million cortical neurons while the human has about 16 billion. It is rare to find a structure that, if multiplied in size by (over) a thousand, retains its basic function without major re-organization. There is one example that comes to mind that has this feature: its basic function is relatively independent of scale, namely, Neural Nets and ‘deep (machine) learning.’

A Neural Net is, in effect, a directed graph, where the vertices—called ‘units’—are thought of as analogous to the neurons in a brain, and each ‘edge’ connecting one vertex to another is assigned a *weight*—a number corresponding to the fluidity of passage along that edge.

There is a battery of nodes in the Neural Net devoted to providing an *input* and a battery of nodes yielding *output* (see Figure 2). For example, a Neural Net might be designed to do the job of ‘reading’ sloppy handwriting: the input would be (the appropriate constellation of bits that are given by) *scribbled numbers* and the desired output would then be (the appropriate constellation of bits that give) the correct typographically *clean number*.

The learning phase of the Neural Net is to iteratively adjust the weights of its edges appropriately so that it does that job better and better. One views the weights on the edges in a Bayesian manner as *priors*, and, given those weights, one then feeds the Neural Net some input and assesses how far the corresponding output is from the correct response. Taking this “assessment of error” as guide, one modifies the weights appropriately in expectation of reducing the size of the error. As the Bayesians say: one ‘educates the priors’ by changing them ever so slightly to produce a slightly more correct output. These changed weights, the *posteriors*, become the priors for the next run-through. The manner in which this ‘education’ occurs is called ‘gradient descent’ which is a

multi-dimensional extension of the classical Newton's method for finding a zero of a differentiable (one variable) function. To be sure, the dimensions related to any useful Neural net (that is, the number of nodes involved) and the number of Bayesian feedback loops required are staggeringly high.

But Neural Nets can do work with as few as mouse-like or as many as human-like numbers of neurons. Mumford writes:

I like to demonstrate in my lectures the way neural nets work with a "Mickey Mouse" example of a neural net with only 12 weights that learns nearly perfectly in front of the live audience to discriminate points in the plane inside a circle from those outside, using simple gradient descent. OpenAI's most recent language program GPT-3 . . . has 175 billion weights and is trained by the same old gradient descent. (pp. 98–99)

5 The World

The title of the Book is *Numbers and The World* and indeed, Mumford thrusts himself into issues, some moving and illuminating, some turbulent, of the world. He begins his Part 6 *Nothing is Simple in the Real World* in a touching remembrance of Klaus and Alice Peters, their publishing company *AKPeters*, their devotedness to "publishing as a service;" and how small math publishers faced the danger of being swallowed by big ones "without an intellectual mission." Burdened by such a threat, Klaus Peters wrote:

Alice and I feel that we have lived a dream to preserve and provide a service that was once considered worthwhile.(p. 190)

Part 6 continues with an admirable, intensely personal account of his experiences in the world—including Japan, the Middle East, Turkey, and India—experiences that gave him "a greater understanding of nationalism; that is, nationalistic governments that "appear to have taken over a large part of the world." (p. 194)

In one chapter Mumford considers whether his Russian mathematical colleague and friend, Igor Shafarevich, should really be viewed as anti-semitic, given the fact that the National Academy of Sciences censured him for "anti-semitic writings and actions" (p. 195) in 1992. This censure was at least in part based on Shafarevich's book *Russophobia*, published in 1989, which is a strident call for national purity and adherence to Russian values, both of which Shafarevich felt were threatened by a (predominantly) Jewish intellectual elite that he called the "little people." Some have called this a "thinly veiled" attack on Jews, while others have sardonically said it "lives up to the best traditions of antisemitic propaganda."

That Shafarevich himself, in a letter to Mumford, denied the technical label of *antisemite* seems irrelevant: his (dangerous) book singles out a list of people, *predominantly Jews*, as blemishing national purity. Throughout history such calls for national purity have had catastrophic consequences.

6 Spinoza, Euclid, Substance and Attributes

In a wonderful chapter, Mumford reflects on his personal shift regarding his thoughts about religion, and his thoughts about Being and Time.

Struggling to find my own path, I stumbled last year upon Spinoza and, to my surprise, found a great deal that I could understand, though not without a struggle. (p. 203)

In his treatise *The Ethics* [3] Spinoza⁸ arms himself for this kind of struggle of thought by framing his argument in a format reminiscent of Euclid's *Elements*—with Definitions, Axioms, and Propositions explicitly stated and with Proofs rooted in this rational structure.

The type of 'struggle of thought' that Mumford experienced may well be intrinsic to any thinking that requires grappling with a view of pantheistic universality. Consider, for example, St. Anselm, a precursor of Spinoza who lived half a millennium before him. Anselm wrote in the Preface to his *Proslogium; Or Discourse on the Existence of God* ([1]):

... one day, when I was exceedingly wearied with resisting its importunity, in the very conflict of my thoughts, the proof of which I had despaired offered itself, so that I eagerly embraced the thoughts which I was strenuously repelling.

This kind of *awkwardness in contemplation* is a useful feature; helpful in dealing with the puzzle and paradox of eternity. Awkwardness is shared in Book XI of Chapter XVI of St. Augustine's *Confessions*⁹ where he reflects;

What, then, is time? If no one asks me, I know what it is. If I wish to explain it to him who asks me, I do not know.

Mumford focusses on Spinoza's notion of *substance*, given by Definition III: that which is in itself, and is conceived through itself: in other words, that of which a conception can be formed independently of any other conception. Mumford brings the word *substance* back to its Greek roots, *ousia*, present participle of *einai* meaning 'being.'

Spinoza's Definition III sets up for his Definition VI: By **God** I mean a being absolutely infinite—that is, a substance consisting in infinite attributes, of which each expresses eternal and infinite essentiality. These definitions lead to Spinoza's Proposition XI: God, or substance, consisting of infinite attributes, of which each expresses eternal and infinite essentiality, necessarily exists.

Mumford explains how Spinoza moves into the classical issue of *mind-and-body*, writing: "God, [Spinoza believed] has infinitely many attributes but only two of them are manifest to our meagre human existences: *extension* and *thought*" (p. 207). Mumford used the term *attribute* as it appears in Spinoza's Definition IV: By **attribute**, I mean that which the intellect perceives as constituting the essence of substance.

On the one hand, it is perfectly reasonable to think of *attribute* as just signifying 'adjectival description;' for example, if you say *cold drink* then 'cold' is an attribute of the drink you had in mind. On the other hand, if, as in Spinoza's definition, an attribute constitutes *the essence of substance*, then should we think of an *attribute* as some kind of 'entity,' a 'thing' in itself?

Mumford continues his discussion of the classical dualism (p. 207):

One should think of these attributes as two of the very many faces of God's essence. The attribute of extension characterizes material objects that exist in space and time.

At this point, Mumford tosses off this striking thought:

In modern physics, we would certainly add "fields"; though non-material, they occupy space and time, so partake of extension. The attribute of thought characterizes all the contents, all the conceptions of our minds. Thus Descartes' mind/body problem is solved by there being two attributes in God's substance. (p. 207)

⁸Benedict de Spinoza (1632-1677).

⁹See also: Eva Brann *What, then, is time?* Paul Dry Books (2001).

It is as if “fields” constitute, so to speak, an *extension* of the realm of the attribute *extension*. Perhaps an intermediary attribute between the material and the noetic.

The concept of ‘field’ emerged on the stage of Physics as an ‘intermediary’ of a different sort. In layman terms, the capabilities of radio transmission puts a strain on the law of conservation of energy: The energy exerted by a radio transmitter is emitted, travels, and in (say) nanoseconds is received as energy by the radio receiver. If one wants to conserve energy one needs to propose some entity that holds this energy during those nanoseconds of transit. Enter: electro-magnetic fields.

James Clerk Maxwell developed this, describing the behavior of electro-magnetism as analogous to hydraulic dynamics, an idea that was fairly immediately appreciated. In his (1855) essay “On Faraday’s Lines of Force” he emphasized the analogy, writing:¹⁰

By a physical analogy I mean that partial similarity between the laws of one science and those of another which makes each of them illustrate the other. . . . [W]e find the same resemblance in mathematical form between two different phenomena.

The notion that fields, of various types, might constitute intermediaries between matter and thought, a kind of merging of matter and thought, is neatly illustrated by Werner Heisenberg’s reflections about the laws of Quantum Mechanics, as quoted by Mumford (p. 160):

We can no longer speak of the behavior of the particle independently of the process of observation. As a final consequence, the natural laws formulated mathematically in quantum theory no longer deal with the elementary particles themselves but with our knowledge of them. Nor is it any longer possible to ask whether or not these particles exist in space and time objectively . . . When we speak of the picture of nature in the exact science of our age, we do not mean a picture of nature so much as a picture of our relationships with nature.

This extraordinary chapter, *Spinoza and Euclid*, was written during the recent pandemic. What a vast span of thought it encompasses. Mumford offers his opinion that “the wall between science and religion is on the verge of crumbling,” (p. 203) and the chapter ends by reciting

“T’was grace that brought us safe thus far, And grace will lead us home.”(p. 213)

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- [4] D’Arcy Wentworth Thompson, *On Growth and Form*, Cambridge University Press, (1917); —there is also a complete revised edition, Dover Press, (1992).

¹⁰This was read to the Cambridge Philosophical Society in two parts, 1855 and 1856. See a discussion, for example, in [2].

- [5] Bernhard Riemann, Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse, Monatsberichte der Berliner Akademie, 1858/60, 671–81. Also in *Gesammelte Mathematische Werke*, 2nd edition, Teubner, Leipzig, 1982, 145–155.
- [6] A.S. Yessenin-Volpin, About Infinity, Finiteness and Finitization (in connection with the Foundations of Mathematics). In *Constructive mathematics*, Las Cruces, NM, 1980, Lecture Notes in Math., **873**(1981), 274–313.

Barry Mazur, Department of Mathematics
Harvard University, Cambridge, MA, USA
E-mail: mazur@g.harvard.edu