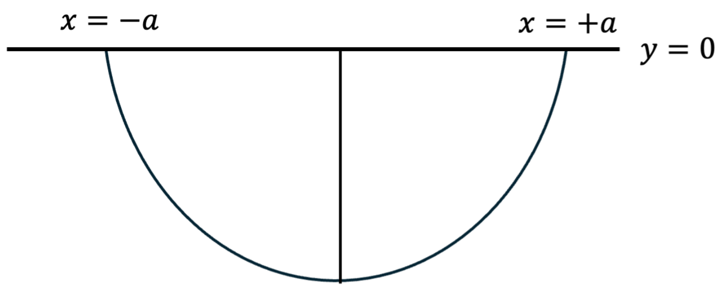
Original Solution by: Edmund Bertschinger
Typeset by: Swati Ravi

Problem 1: Catenary



(a) **Derivation:**

The equilibrium shape is the one that minimizes the potential energy:

$$U = g \int \mu y \, d\ell = \mu g \int_{-a}^{a} y\sqrt{1+y'^2}\, dx, \quad \text{where } y' = \frac{dy}{dx}.$$

Subject to the constraint:

$$\int_{-a}^{a} d\ell = \int_{-a}^{a} \sqrt{1+y'^2}\, dx = \ell.$$

Using the Lagrange multiplier to enforce the constraint, we define:

$$I[y(x)] = U - \lambda\ell = \int_{-a}^{a} (\mu g y - \lambda)\sqrt{1+y'^2}\, dx.$$

This is the functional $I[y(x)]$ whose extremum gives the equilibrium shape of the chain as the curve $y(x)$.

(b) Let $L(y, y'; x) = (\mu g y - \lambda)\sqrt{1+y'^2}$.

Then,

$$\delta I = 0 \rightarrow \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = 0$$

$$\frac{d}{dx}\left[\frac{(\mu g y - \lambda)y'}{\sqrt{1+y'^2}}\right] - \frac{\mu g}{\sqrt{1+y'^2}} = 0$$

λ will be determined by enforcing the constraint.

Given solution:

$$y = A\cosh(kx+\phi) + B$$

$$y' = kA\sinh(kx+\phi), \quad \sqrt{1+y'^2} = \sqrt{1+k^2A^2\sinh^2(kx+\phi)}$$

$$\frac{d}{dx}\left[\frac{(\mu g y - \lambda)y'}{\sqrt{1+y'^2}}\right] = \frac{\mu g}{\sqrt{1+y'^2}}$$

$$\frac{d}{dx}\left[\frac{\mu g A\cosh(kx+\phi)+\mu g B-\lambda}{\sqrt{1+k^2A^2\sinh^2(kx+\phi)}}\right]=\frac{\mu g}{\sqrt{1+k^2A^2\sinh^2(kx+\phi)}}$$

Satisfied when:

$$kA=1,\quad \mu g B=\lambda \quad\rightarrow\quad \sqrt{1+y'^2}=\cosh(kx+\phi)\quad \text{and}$$

$$\frac{d}{dx}\left[\mu g A\sinh(kx+\phi)\right]=\mu g\cosh(kx+\phi)$$

Therefore,

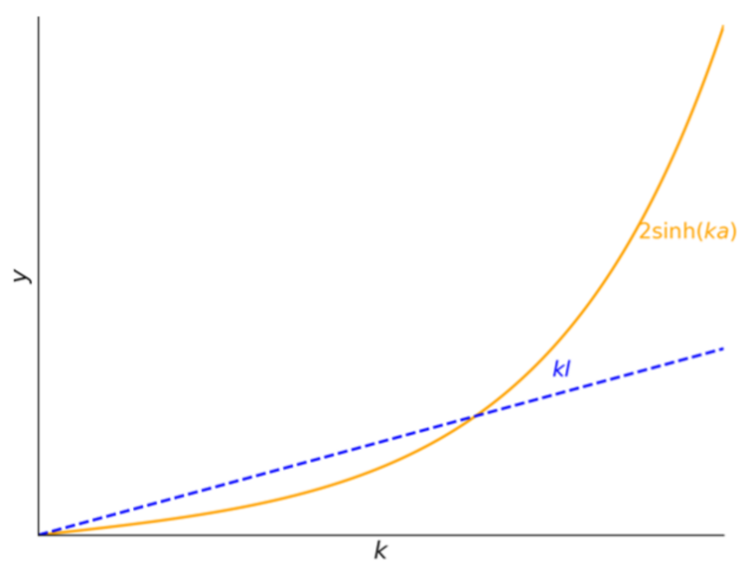
$$A=k^{-1},\quad B=\frac{\lambda}{\mu g},\quad \text{where } k,\lambda \text{ remain to be determined.}$$

Enforcing boundary conditions at $x=\pm a$: $y=0$,

$$0=k^{-1}\cosh(\pm ka+\phi)+\frac{\lambda}{\mu g}-\phi=0;\quad k^{-1}\cosh(ka)=-B=-\frac{\lambda}{\mu g}$$

We get λ by enforcing the constraint:

$$\lambda=\int_{-a}^{a}\cosh(kx)dx=2k^{-1}\sinh(ka)=\ell$$



The last equation is an implicit equation for $k(\ell)$. As the sketch shows, it has a unique positive root. Once k is known, B,λ follow from $B=\frac{\lambda}{\mu g}=-k^{-1}\cosh(ka)$ and $A=k^{-1}$.

(c) **Tension on the chain T(x):**

The tension follows by considering adding a section of length $d\ell$ to the chain at some location (x,y). The potential energy of the chain increases by $\mu g y d\ell$, and making a gap of size $d\ell$ releases energy $Td\ell$. Thus, the change in potential energy of the chain is $(\mu g y-T)d\ell$. But if the chain remains a catenary (minimum potential energy), then:

$$dI=\left(\frac{\partial I_{\min}}{\partial \ell}\right)d\ell=\lambda d\ell=(\mu g y-T)d\ell \quad \text{where we have used the result from part (a).}$$

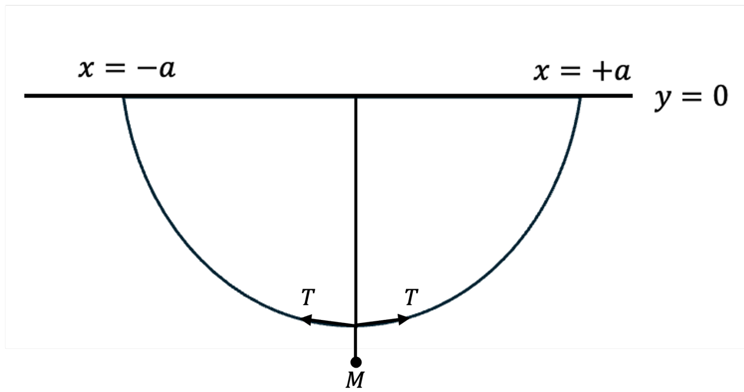
$$T(x)=\mu g y(x)-\lambda=\mu g[y(x)-B]=\frac{\mu g}{k}\cosh(kx).$$

(d) **Adding a mass M:**

Either apply Newton's law $\Sigma F = 0$ at the mass, or modify the variational principle. We will show the latter.

$$I[y(x)] = \int_{-a}^{a} L(y)\,dx + Mgy(0); \quad L(y) = (\mu g y - \lambda)\sqrt{1+y'^2}.$$

We expect y' to be discontinuous at $x = 0$, so the variational principle must exercise care at $x = 0$.



$$y(x) \to y(x) + \delta y(x) \quad \text{(small perturbation)}.$$

$$\delta I = \int_{-a}^{0} \left[\frac{\partial L}{\partial y'}\delta y'(x) + \frac{\partial L}{\partial y}\delta y(x)\right] dx + \int_{0}^{a} \left[\frac{\partial L}{\partial y'}\delta y'(x) + \frac{\partial L}{\partial y}\delta y(x)\right] dx + Mg\delta y(0).$$

Integrating by parts:

$$\delta I = \int_{-a}^{0} \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right] \delta y(x)\,dx + \int_{0}^{a} \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right] \delta y(x)\,dx + \left[\frac{\partial L}{\partial y'}\delta y(x)\right]_{x=0^-}^{x=0^+} + Mg\delta y(0).$$

Since $\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = 0$ as before for $x \neq 0$, and:

$$Mg = \left[\frac{\partial L}{\partial y'}\right]_{x=0} \quad \text{(the jump at } x = 0).$$

Now we know from (b) that $y(x) = A\cosh(kx+\phi) + B$ satisfies the Euler–Lagrange equation if $A = k^{-1}$ and $B = \frac{\lambda}{\mu g}$. For boundary conditions:

$$y = 0 \quad \text{at } x = \pm a \quad \to \quad k^{-1}\cosh(\pm ka + \phi) = -B.$$

Let:

$$\phi_+ = -\phi_- = \phi_0 \quad \to \quad k^{-1}\cosh(ka + \phi_0) = -B = -\frac{\lambda}{\mu g} \quad \ldots(3).$$

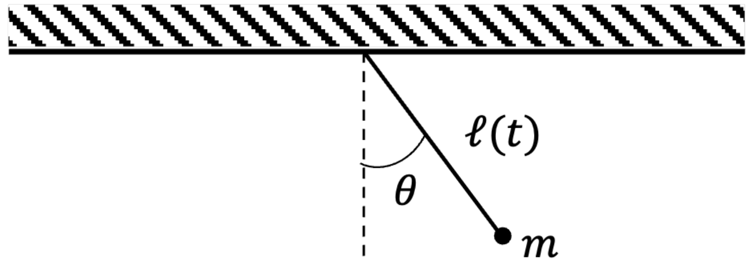
$$\left[\frac{\partial L}{\partial y'}\right]_{x=0} = Mg = \frac{\mu g}{k}\left[\sinh(\phi_0) - \sinh(-\phi_0)\right] \to \sinh(\phi_0) = \frac{kM}{2a} \quad \ldots(1).$$

Finally:

$$\frac{\ell}{2} = \int_0^a \sqrt{1+y'^2}\,dx = \int_0^a \cosh(kx+\phi_0)\,dx = k^{-1}\left[\sinh(ka+\phi_0) - \sinh(\phi_0)\right] = \frac{\ell}{2} \quad \ldots(2).$$

Solve (1) and (2) simultaneously for k, ϕ_0, then solve (3) for B and (4) for A.

Problem 2: Plane Pendulum



a) **Lagrangian:** $L = T - V$

$$T = \frac{1}{2}m\left(\dot{\ell}^2 + \ell^2\dot{\theta}^2\right), \quad V = -mg\ell\cos\theta$$

$$L = \frac{1}{2}m\left(\dot{\ell}^2 + \ell^2\dot{\theta}^2\right) + mg\ell\cos\theta$$

Canonical Momentum:

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = m\ell^2\dot{\theta}, \quad \dot{\theta} = \frac{p_\theta}{m\ell^2}$$

Hamiltonian:

$$H = \dot{\theta}p_\theta - L = \frac{1}{2}\frac{p_\theta^2}{m\ell^2} + \frac{1}{2}m\dot{\ell}^2 - mg\ell\cos\theta$$

b) **Energy:**

$$E = T + V = \frac{1}{2}m\left(\dot{\ell}^2 + \ell^2\dot{\theta}^2\right) - mg\ell\cos\theta = H + m\dot{\ell}^2 \neq H$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \neq 0 \quad \text{if } \dot{\ell} \neq 0 \quad \rightarrow \quad H \text{ is not conserved.}$$

$$\frac{dE}{dt} = \frac{\partial H}{\partial t} + 2m\dot{\ell}\ddot{\ell} - \frac{p_\theta^2}{m\ell^2}\frac{\dot{\ell}}{\ell} + m\ddot{\ell}\dot{\ell} - mg\dot{\ell}\cos\theta \neq 0 \quad \text{in general} \quad \rightarrow \quad E \text{ is not conserved.}$$

Work must be done to change ℓ given the tension in the string. If the Hamiltonian and the total energy are distinct, the total energy is not conserved.

c) **Equations of Motion:**

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\ell^2}, \quad p_\theta = -\frac{\partial H}{\partial\theta} = -mg\ell\sin\theta$$

$$\frac{d}{dt}\left(m\ell^2\dot{\theta}\right) + mg\ell\sin\theta = 0 \quad \rightarrow \quad \ddot{\theta} + 2\frac{\dot{\ell}}{\ell}\dot{\theta} + \frac{g}{\ell}\sin\theta = 0 \quad \text{(Second order ODE).}$$

Special Case: When $\dot{\ell} = 0$, the above ODE reduces to:

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

Small Oscillations: For small θ, $\sin\theta \approx \theta$, leading to:

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0$$

Solution: For $\theta^2 \ll 1$:

$$\theta \propto \cos(\omega t + \phi), \quad \text{where } \omega = \sqrt{\frac{g}{\ell}}$$

d) **Adiabatic Invariant:**

$$\left(\frac{\dot{\ell}}{\ell}\right)^2 \ll \omega^2, \quad I_\theta = \int p_\theta \, d\theta \quad \text{is an adiabatic invariant.}$$

$$I_\theta = \int m\ell^2 \dot{\theta} \, d\theta \approx m\ell^2 \int_0^{2\pi/\omega} \dot{\theta}^2 \, d\theta$$

Ansatz: $\theta = A(t)\cos(\omega t + \phi)$ where $\left(\frac{\dot{A}}{A}\right)^2 \ll \omega^2$ (slowly varying amplitude).

$$I_\theta = m\ell^2 A^2 \omega^2 \frac{1}{2} \cdot \frac{2\pi}{\omega} = \pi m\ell^2 \omega A^2 = \pi m\ell^2 \sqrt{\frac{g}{\ell}} A^2 \propto \ell^{3/2}(t) A^2(t) = \text{constant.}$$

$$A^2(t) \propto \ell^{-3/2}(t) \quad \text{or} \quad A(t) \propto \ell^{-3/4}(t) \quad \text{Q.E.D.}$$

We can also obtain this result by applying the WKB method to the differential equation.