LECTURE 1 Mon 9/4

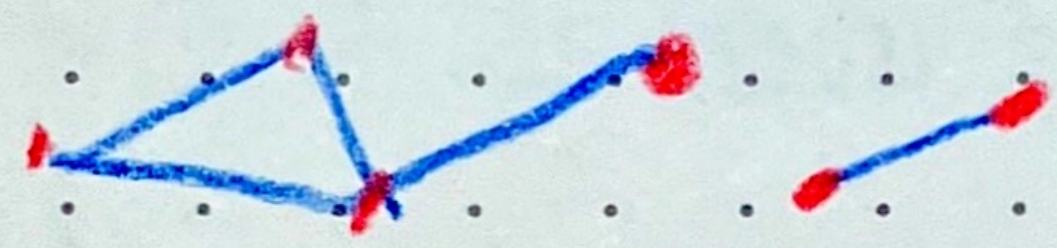
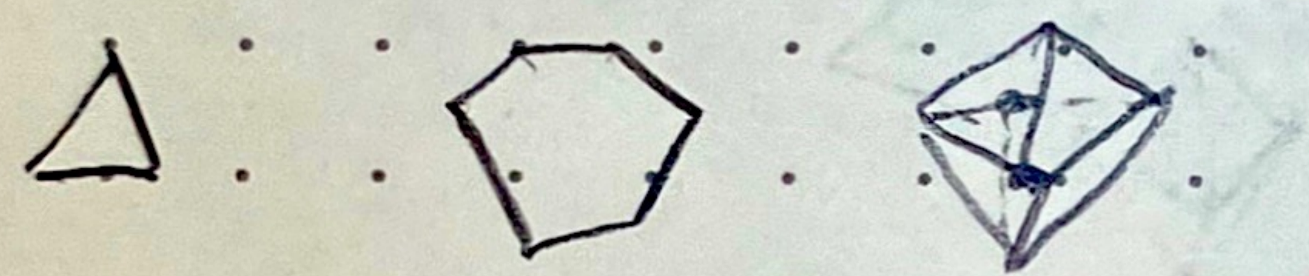
Combinatorics: How to count?
Studies <u>discrete</u> objects

Geometry: Usually <u>continuous</u>

We use combinatorics as a perspective to think about any type of problem. Hooray Ü

Similarly geometry can be a way about thinking about math problems. (continuously)

<u>Main Players</u>:

- graphs (V, E) — V: vertices, E: edges 
- polytopes 
- hyperplane arrangements 
- matroids
- etc

Previous Related Courses
- Spring 2024 Lauren Williams' Class
- Spring 2020 (Postnikov) Polytopes & Hyperplane arrangements (Notes online)
- 2016(?)(Postnikov) Polytopes
- 2004(?) Richard Stanley's class

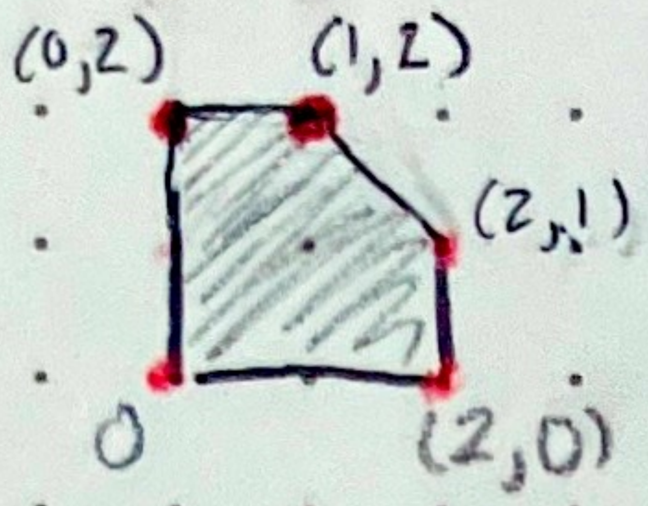
graphs → hyperplane arrangements
graphs → (graphical) matroids
hyperplane arrangements ↓ zonotopes
matroids ↔ polytopes ⊃ zonotopes

Basically, all these things are very related.

Intro to (convex) polytopes

$P \subset \mathbb{R}^d$

E.x.



5 vertices
5 edges
1 2-dim face.

<u>Def</u>: <u>Face numbers</u>

$f_i = \#\, i$-dim'l faces of P

<u>f-vector</u> $(f_0, f_1, \ldots, f_d)$ for dim d polytope

<u>f-polynomial</u> $f(x) = \sum_{i=0}^{d} f_i x^i$

<u>E.x.</u> For polytope above $f(x) = 5 + 5x + x^2$

<u>Def</u>: <u>h-polynomial</u>

$h(x) = f(x-1) = \sum_{i=0}^{d} h_i x^i$

<u>h-vector</u> : $(h_0, h_1, \ldots, h_d)$

<u>E.x.</u> From above

$h(x) = 5 + 5(x+1) + (x+1)^2$

$= 1 + 3x + x^2$

h-vector: $(1, 3, 1)$

<u>Thrm</u>: For a simple d-dim polytope P,

- $h_0, h_1, \ldots$ are positive integers

- <u>They are palendromic</u>

i.e. $\boxed{h_i = h_{d-i} \ \forall i}$ Dehn-Somerville equations

Valuations of P

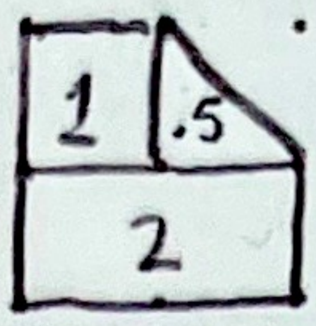
Vol(P) (volume)

lattice pts $= \#(P \cap \mathbb{Z}^d)$

Ehrhart polynomial

E.x. (cont'd)

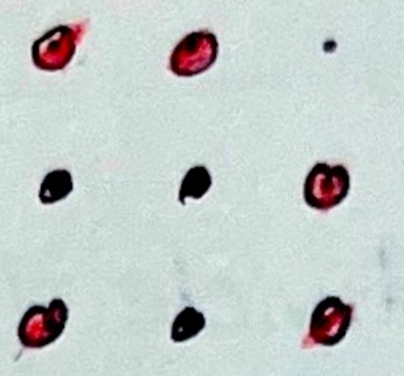
$\mathrm{Vol}(P) = 3.5 = \frac{1}{2} \cdot 7$



Lemma: $\forall$ integer d-dim polytope,

$\underbrace{d!\,\mathrm{Vol}(P)}_{\text{normalized volume}} \in \mathbb{Z}$

E.x. # lattice pts = 8



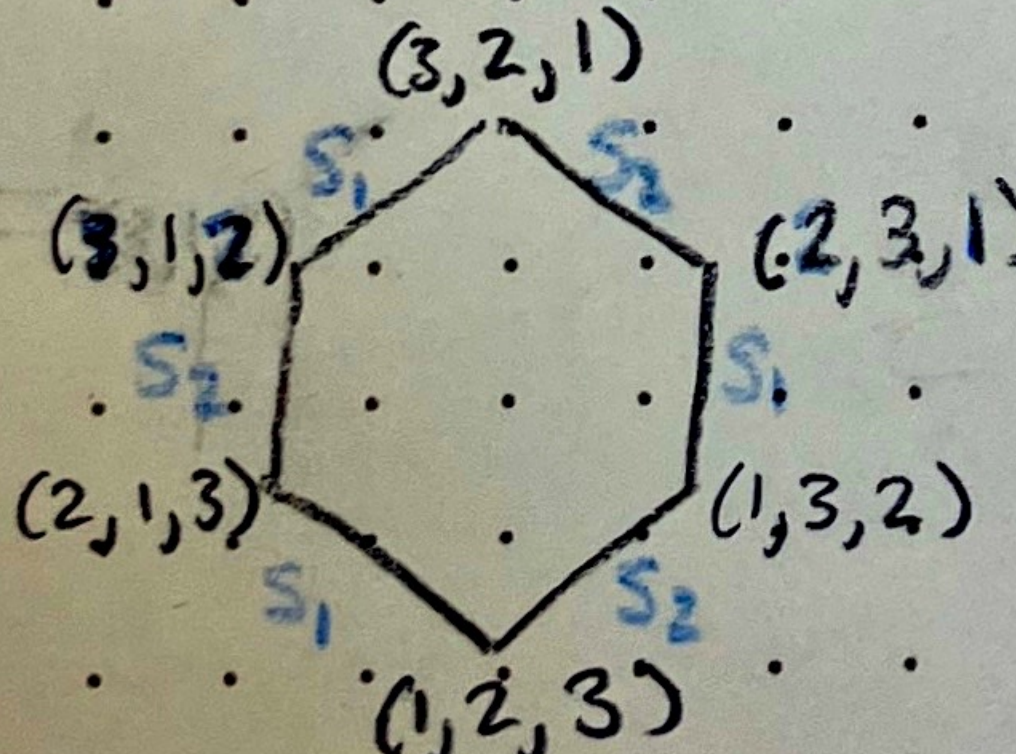
Many interesting properties in other areas of math can be expressed in terms of these valuations of polytopes.

Some combinatorially significant polytopes:

Def: Permutohydron (Permutahedron)

$\Pi_n = \mathrm{conv}\left((w_1, \ldots, w_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} w_1, \ldots, w_n \text{ is a} \\ \text{permutation of } 1, 2, \ldots, n \end{array} \right\}$

E.x. $\Pi_3 =$



$\subset$ $x+y+z=6$ hyperplane of $\mathbb{R}^3$

Note: Edges correspond to transpositions of adjacent values (not positions).

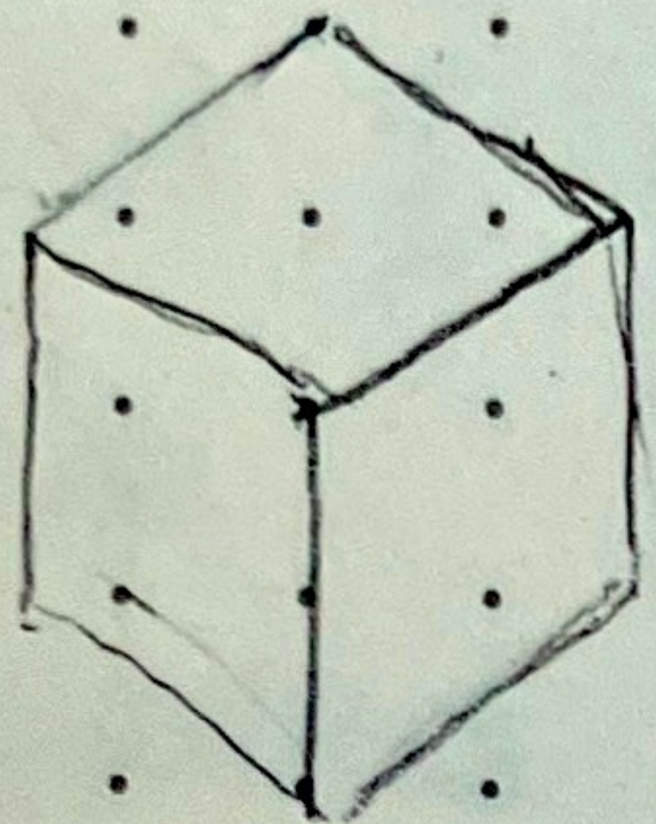
Some Properties –

- $\dim \Pi_n = n-1$
- # vertices $= n!$
  - → No vertices can be on the inside, because then by symmetry they would all have to be
- $f_{n-i}(\Pi_n) = i!\,S(n,i)$ ← Stirling # of the 2nd kind

- $h_i(\Pi_n)$ = Eulerian numbers
  = (# permutations of $1, \ldots, n$ w/ exactly i descents)

- $\mathrm{Vol}\,\Pi_n = n^{n-2}$ = (Cayley's formula for # labelled trees on n vertices)

in Ex.

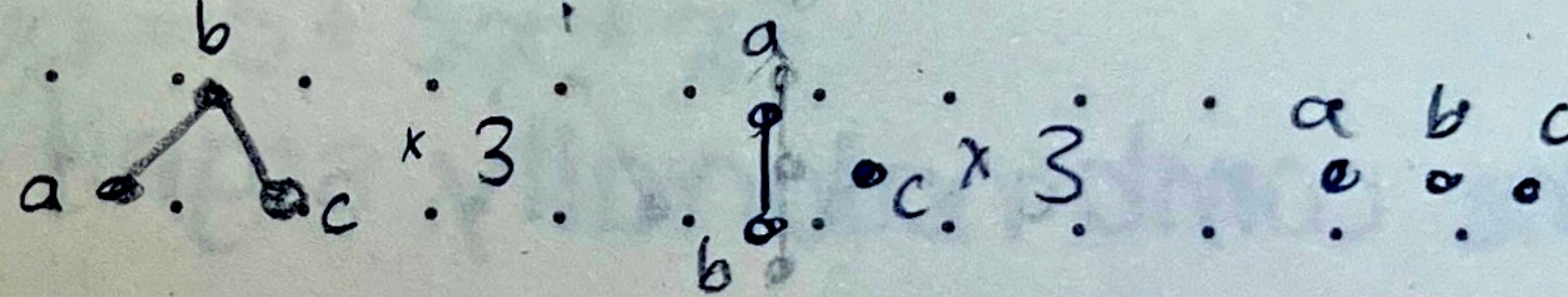


Each piece has volume 1

In general can subdivide into parallel pipeds of volume 1, that are in bijection w/ labelled trees on n vertices (will show later)

- # lattice points = $\#(\Pi_n \cap \mathbb{Z}^n)$
  = # labelled forests on n nodes

  ↳ For Π_3 we get 7



<u>Challenge Problem</u>: Prove last bullet point bijectively