LECTURE 3 Mon 9/9

__Last Time__: Convex polytopes $P \subset \mathbb{R}^n$

$$P = \text{conv}(v_1, \ldots, v_m) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} \leq \vec{b}\}$$

polar duality

__faces of P__ = supporting faces $F_{a,b}$ of P for some linear fcn $a(x) \in (\mathbb{R}^n)^*$

__vertices__ = 0-dim faces

__edges__ = 1-dim faces

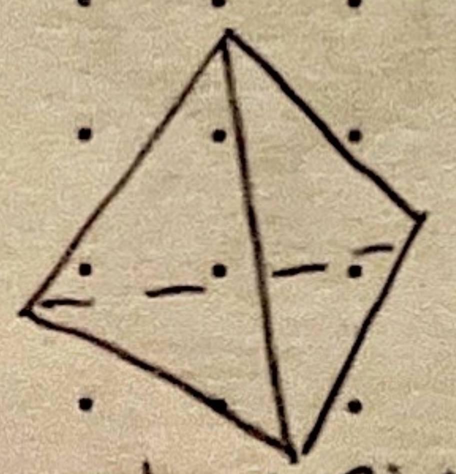
__facets__ = (d-1)-dim faces $d = \dim(P)$

Simple vs. Simplicial Polytopes

__Def 1__: A d-dim polytope is __simple__ if every vertex is incident to exactly d edges

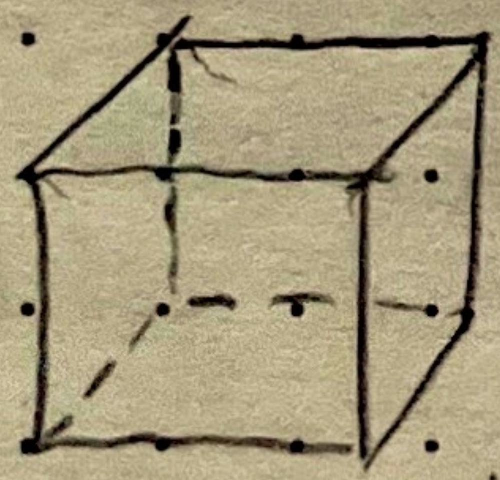
__Def 2__: P is __simplicial__ if every facet of P is a (d-1)-symplex.

__Ex.__



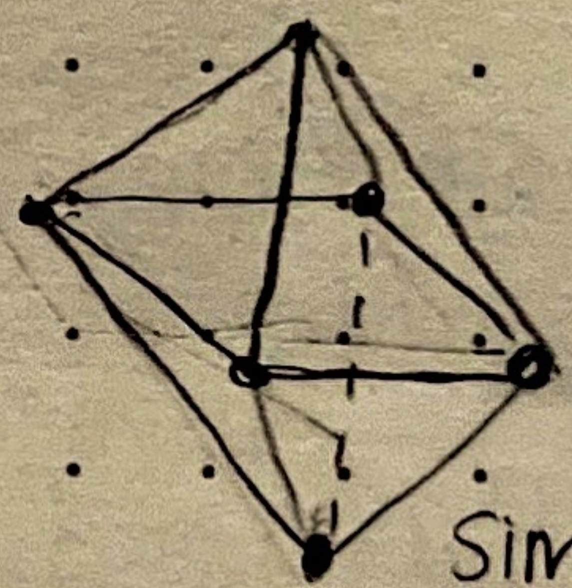
Its own polar dual

simple & simplicial

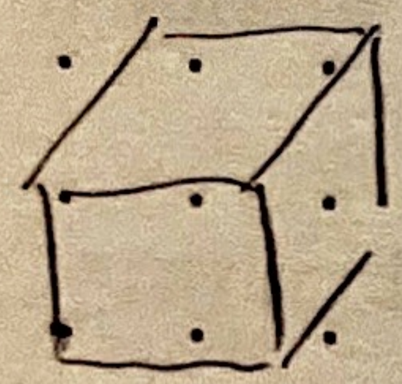


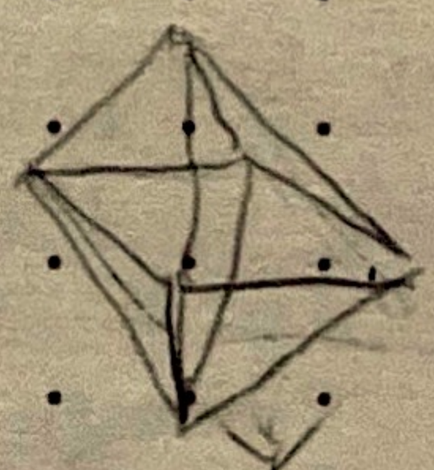
simple only

Polar duals



Simplicial only





$\begin{cases} \pm x \leq 1 \\ \pm y \leq 1 \\ \pm z \leq 1 \end{cases}$ $\xleftrightarrow{\text{Polar dual}}$ $\text{conv}\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$

__Def__: __Combinatorial type__ of P = the face poset of P (ordered by inclusion)

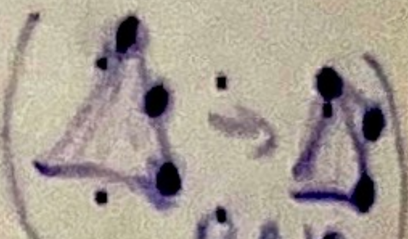
Prop: Assume that $P \subset \mathbb{R}^n$ and $\dim P = n$.

(1). $P = \{\vec{x} \mid A\vec{x} = \vec{b}\}$ is simple iff it has the same combinatorial type as the polytope $P' = \{\vec{x} \mid A'\vec{x} \leq \vec{b}'\}$ for all $(A', \vec{b}')$ sufficiently close to $(A, \vec{b})$.

Must be min. size of A

(2). $P = \mathrm{conv}(v_1, \dots, v_m)$ is simplicial iff it has the same comb type as $P'' = \mathrm{conv}(v_1'', \dots, v_m'')$ for all $v_1'', \dots, v_m''$ sufficiently close to $v_1, \dots, v_m$.

← must actually minimal set of v_i



BAD

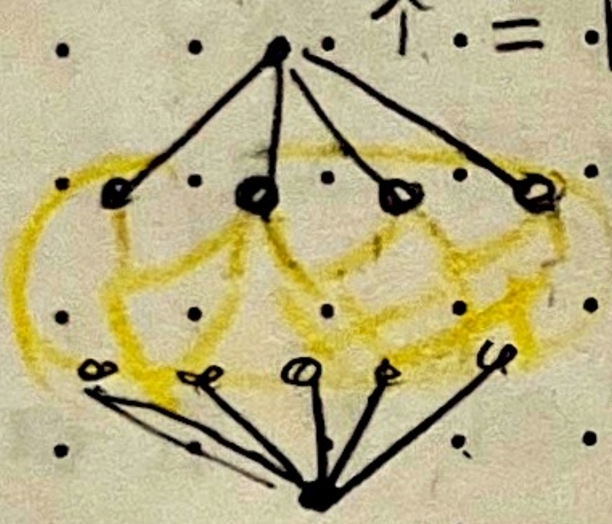
(i.e.) For (1) Moving the bounding hyperplanes keeps it simple but moving vertices can make it change.

(2) Other way around

Arguments for & against counting empty face as a face.

Face poset of P

$\hat{1} = F$



co-atoms = facets

atoms = vertices

$\hat{0} = \emptyset$

Lemma: P & P^* have dual face posets
i.e. flip P upsidedown to get P^*

$$f_i(P) = f_{d-1-i}(P^*) \qquad -1 \leq i \leq d \qquad f_{-1} = f_d = 1$$

Simplicial polytopes → simplicial complexes
In this case, $\emptyset$ should be a face but P is not

↳ f-vector $(f_{-1}, f_0, \dots, f_{d-1})$

Simple polytopes
$\emptyset$ is NOT a face but P is
f-vector $(f_0, f_1, \dots, f_d)$

↳ By default we will use this convention

Why this convention?
- will be nice to start at 0 not -1
- Get $h(x) = f(x-1)$ instead of needing some shift by binomial coeffs.

Assume: P is simple.

f-vector $(f_0, \dots, f_d)$ $\quad f(x) = \sum_{i=0}^{d} f_i x^i$

h-vector $(h_0, \dots, h_d)$ $\quad h(x) = \sum_{i=0}^{d} h_i x^i$

h's defined by $\boxed{h(x) = f(x-1)}$

h_i's all non-neg $\rightsquigarrow$ Should have combinatorial interpretation

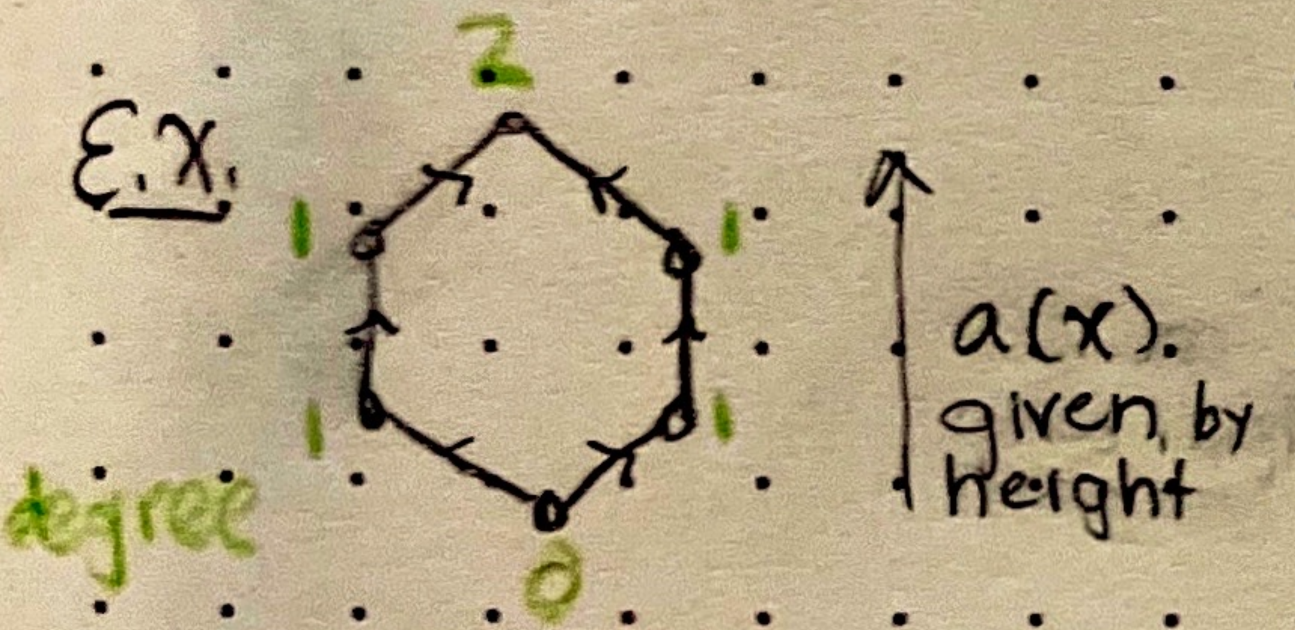
Fix a generic lin. function $a(x)$, ⟵ different value on each vertex

& orient 1-skeleton of P ⟵ take just the vertices & edges, forget the rest

$\underset{u}{\bullet} \longrightarrow \underset{v}{\bullet}$ if $a(u) < a(v)$.

Thrm: $h_i(P) = \#$ vertices of P with degree i

Ex.



f-vector $(6, 6, 1)$
h-vector $(1, 4, 1)$
$h_0 = 1, \; h_1 = 4, \; h_2 = 1$

Toric varieties

$P \rightsquigarrow X_P$

moment map : $X_P \longrightarrow P$

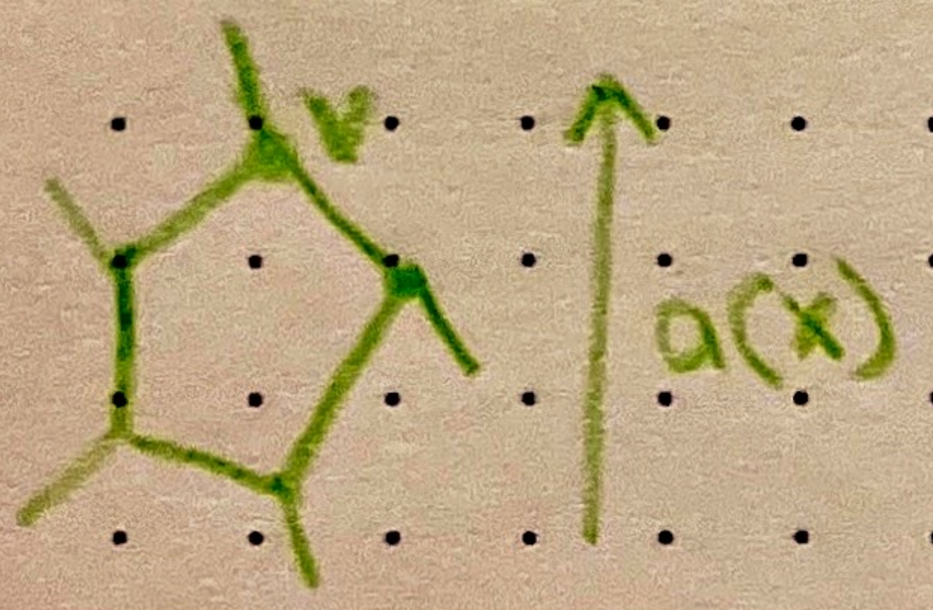
Generic lin. fcn on $P \rightsquigarrow$ A Morse function on X_P

Morse Theory $\Rightarrow \dim H^{2i}(X_P) = h_i(P)$.

Motivation of h #'s
Will get to some of this properly later

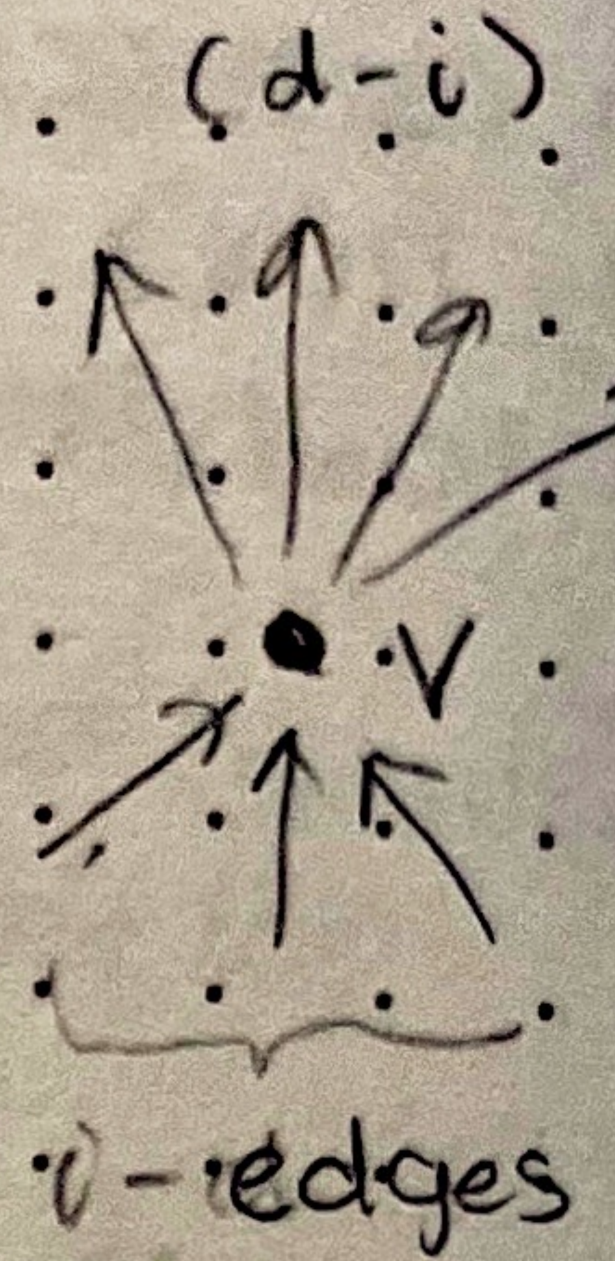
Proof of Thrm: The map $M: \begin{Bmatrix}\text{face} \\ \text{of } P\end{Bmatrix} \longrightarrow \begin{Bmatrix}\text{vertices} \\ \text{of } P\end{Bmatrix}$

$F \longmapsto \underbrace{v = F_{a,F}}_{\text{a vertex where } a \text{ takes its max value}}$



Fix a vertex v with in-degree i

$$\sum_{\substack{F \text{ faces s.t.}\\ M[F]=v}} x^{\dim F} = \underbrace{1}_{F=v} + \underbrace{ix}_{\substack{F=\text{one}\\ \text{of the}\\ i \text{ edges}}} + \underbrace{\binom{i}{2}x^2}_{\substack{F=\text{face}\\ \text{given by 2}\\ \text{of the } i\\ \text{edges}}} + \dots = (x+1)^i$$



Note: To make this part rigorous requires an additional lemma not addressed here.

$$f(x) = \sum_F x^{\dim F}$$

$$= \sum_{v \text{ vertex}} \Big(\sum_{\substack{F \text{ s.t.}\\ M(F)=v}} x^{\dim F} \Big)$$

$$= \sum_{v \text{ vertex}} (x+1)^{\text{in-degree}(i)} = h(x+1)$$

$\Rightarrow h_i = \#$ vertices of in-degree i

<u>Cor</u>: $h_i(P) = h_{d-i}(P)$

<u>Proof</u>: If you instead of using function a use $-a$, then the whole graph reverses, so the values for h_i & h_{d-i} flip
$\Rightarrow$ must be the same