LECTURE 9 Wed 9/25

Last Time: regular zonotopal tilings

zonotope $Z = \sum_{i=1}^{N} [0, \vec{v}_i] \subset \mathbb{R}^d$

"lifted" zonotope $\tilde{Z} = \sum_{i=1}^{N} [0, \tilde{v}_i] \subset \mathbb{R}^{d+1}$

$\tilde{v}_i = (\vec{v}_i, h_i) \subset \mathbb{R}^{d+1}$

("heights)

$p: \mathbb{R}^{d+1} \to \mathbb{R}^d$
$p(\tilde{Z}) = Z$

The regular zonotopal tiling (assosciated w/ $h_1, \ldots, h_N$) is the "projection of the upper boundary" of $\tilde{Z}$ onto Z

Tiles: $p(F_{\tilde{a}, \tilde{Z}}) = \sum_{i=1}^{N} \begin{cases} \{\vec{v}_i\} & \text{if } \langle \vec{a}, \vec{v}_i \rangle + h_i > 0 \\ \{0\} & \text{if } \langle \vec{a}, \vec{v}_i \rangle + h_i < 0 \\ [0, \vec{v}_i] & \text{if } \langle \vec{a}, \vec{v}_i \rangle + h_i = 0 \end{cases}$ here $\vec{a} \in \mathbb{R}^d$

Affine hyperplane arrangement

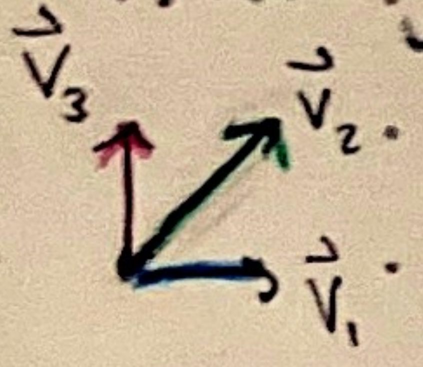
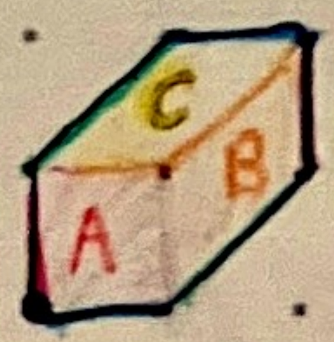
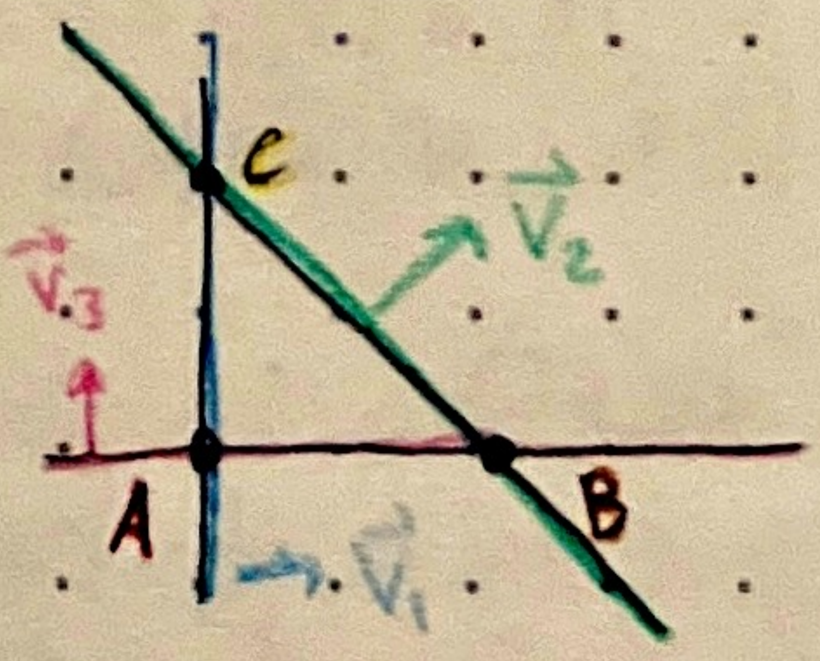
$\mathcal{A}' = \{H_1', \ldots, H_N'\}$

$H_i' = \{\vec{a} \in \mathbb{R}^d \mid \langle \vec{a}, \vec{v} \rangle + h_i = 0\}$

Thrm: For any choice of heights, the face poset of $\mathcal{A}$ is dual to the face poset of the assosciated zonotopal tiling

All distinct, in this case

Thrm: If h_i's are in general position, the the corresponding tiling is fine

Ex.  $Z =$  $\mathcal{A} =$ 

Tiles:

A: $[0, \vec{v}_1] + \{0\} + [0, \vec{v}_3]$

B: $\{\vec{v}_1\} + [0, \vec{v}_2] + [0, \vec{v}_3]$

C: $[0, \vec{v}_1] + [0, \vec{v}_2] + \{v_3\}$

what lines does it intersect (intervals)
and what height along direction it doesn't (points)

Lemma: For h_i's in general position,

$$H_{i_1} \cap \dots \cap H_{i_k} \neq \emptyset \text{ iff } \vec{v}_{i_1}, \dots, \vec{v}_{i_k} \text{ lin. ind.}$$

A generalization: Mixed subdivisions of Minkowski sum of polytopes
$P = P_1 + P_2 + \dots + P_N \subseteq \mathbb{R}^d$

Def: A mixed subdivision of P is a collection of labelled tiles of the form ✓ labelled by $(F_1, \dots, F_N)$

$$T_{(F_1, \dots, F_N)} = F_1 + \dots + F_N$$

where F_i is a face of P_i $\forall i$, and satisfying

(0) Each tile is d-dim'l

(1) Union of tiles $= P$

(2) Any pair of tiles intersect properly:

$$T_{(F_1, \dots, F_N)} \cap T_{(G_1, \dots, G_N)} = T_{(F_1 \cap G_1, \dots, F_n \cap G_n)};$$

the common face of the two tiles. (or $\emptyset$)

Def: Regular mixed subdivision:

$p: \mathbb{R}^{d+1} \to \mathbb{R}^d$

Lifted polytopes $\tilde{P}_1, \dots, \tilde{P}_n \subset \mathbb{R}^{d+1}$ s.t.

$p: \tilde{P}_i \to P_i$ is a linear isomorphism.

$\tilde{P} = \tilde{P}_1 + \dots + \tilde{P}_N$

$\rightsquigarrow$ Mixed subdivision of P

= Projection of the upper bdry of $\tilde{P}$

(b/c p gives lin. iso, projections of faces of $\tilde{P}_i$ go to faces of P_i).

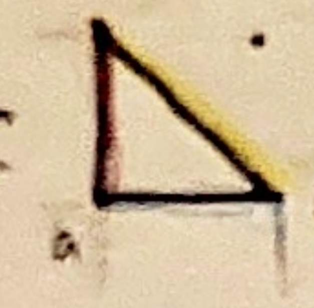
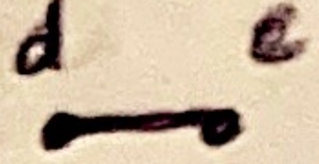
Suppose $\mathcal{N}_i = \mathcal{N}_{P_i}$, normal fan of P_i

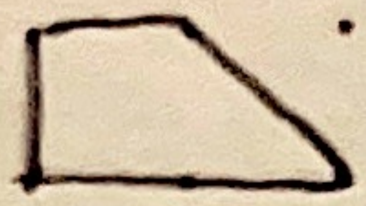
$$\mathcal{N}_i' = \mathcal{N}_{P_i} + \vec{u}_i$$

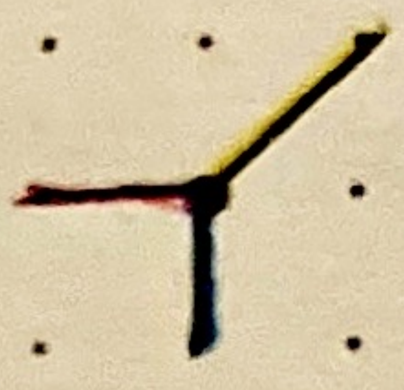
Affine fan arrangement

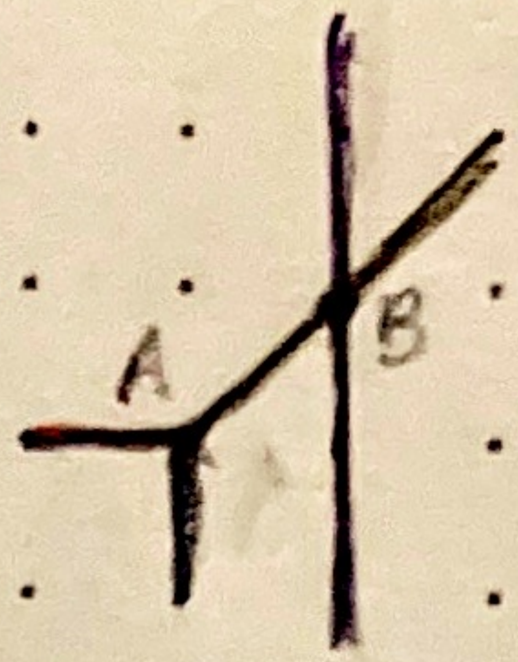
$$\mathcal{A}' = \{\mathcal{N}_1', \dots, \mathcal{N}_N'\}$$

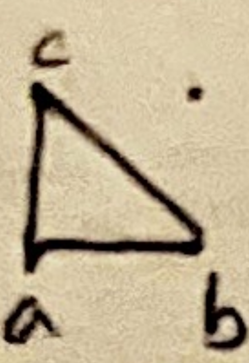
Thrm: The face poset of $\mathcal{A}'$ is dual to the face poset of the corresponding mixed subdivision of P

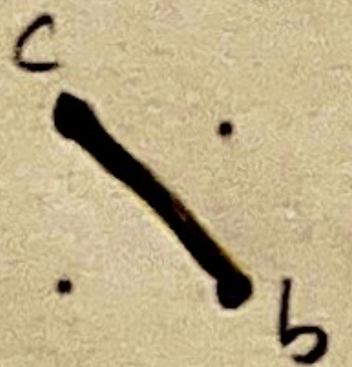
Ex. $P_1 =$  $P_2 =$ 

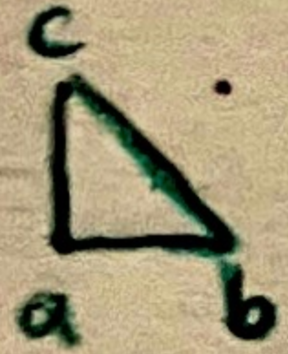
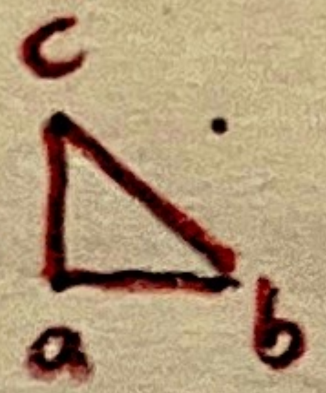
$P = P_1 + P_2 =$ 

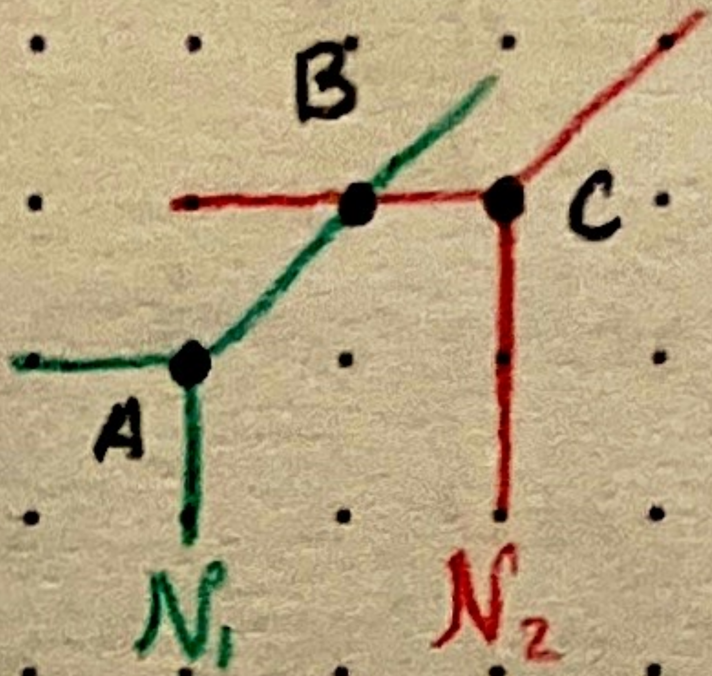
$N_1 =$  $N_2 =$ 

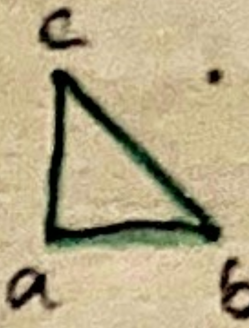
$A' =$ 

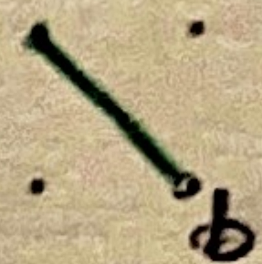
Tiles A :  + $\{d\}$

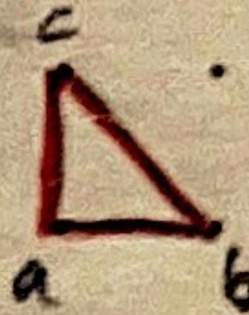
B :  + 

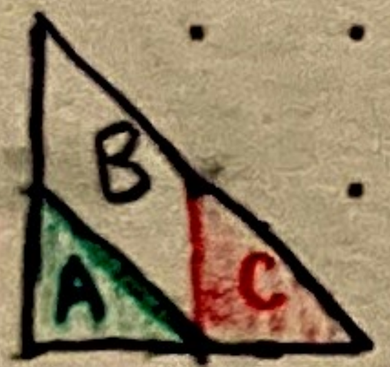
Ex. $P_1 =$  $P_2 =$ 



$A =$  + $\{a\}$

$B =$  + 

$C = \{b\} +$ 

$P = P_1 + P_2 =$  (labelling matters)

Choice of $\tilde{P}_i$ corresponds to choice of lin. fcn. on $\mathbb{R}^d$ which correspond to our choice of $\vec{u}_i$ in $\tilde{N}_i = N_i + \vec{u}_i$

Lemma: For $\vec{u}_i$'s in general position, each tile is of the form $F_1 + \dots + F_N$ where $\operatorname{span} F_1 \oplus \dots \oplus \operatorname{span} F_N = \mathbb{R}^d$ is a decomposition of $\mathbb{R}^d$ as direct sum of subspace

$\Rightarrow$ Thm from last lecture that volume of dilated polytopes is polynomial of t_is