LECTURE 11 MON 9/30

Mixed Volume

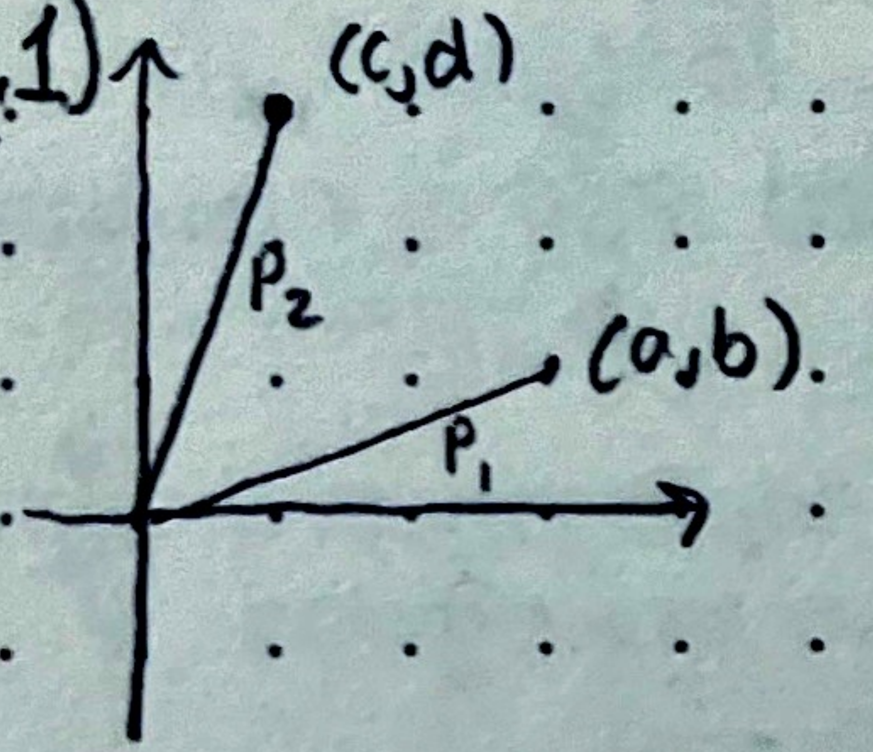
$P_1, \ldots, P_d \subset \mathbb{R}^d$ polytopes. (don't need to be full dim'l)

Know: $\mathrm{vol}(t_1 P_1 + \ldots + t_d P_d)$ is polynomial in t_i's

Def: The mixed volume of $P_1, \ldots, P_d$ is

$$V(P_1, \ldots, P_d) = \frac{1}{d!}\left(\text{coeff of } t_1 t_2 \cdots t_d \text{ in } \mathrm{Vol}(t_1P_1 + \ldots + t_dP_d)\right)$$

Ex. 1)

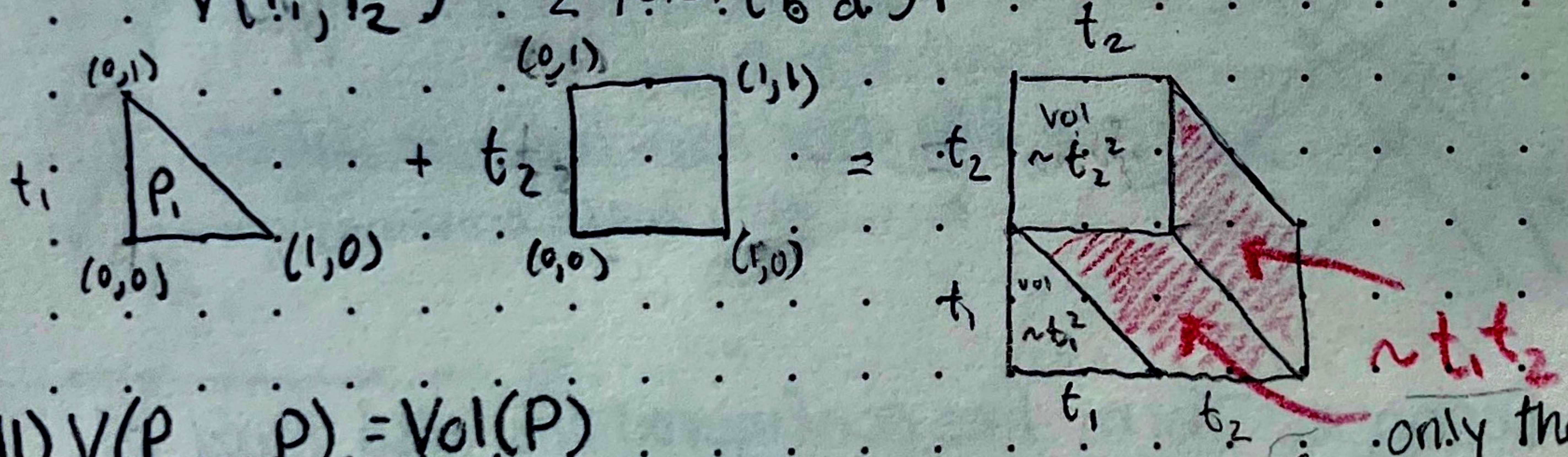


$P_1 = [\vec{0}, (a,b)]$ $P_2 = [\vec{0}, (c,d)]$

$\mathrm{Vol}(t_1 P_1 + t_2 P_2) = \left| \det \begin{bmatrix} t_1 a & t_2 c \\ t_1 b & t_2 d \end{bmatrix} \right|$

$$V(P_1, P_2) = \frac{1}{2} \left|\det\begin{pmatrix} a & c \\ b & d\end{pmatrix}\right|$$

Ex. 2)



only these (coming from mix of both polytopes) contribut to mixed volume)

Prop: (A1) $V(\underbrace{P, \ldots, P}_{d}) = \mathrm{Vol}(P)$

(A2) $V(P_1, \ldots, P_d)$ is symmetric w.r.t. perms of P_i's.

(A3) $V(P_1, \ldots, P_d)$ is a multilinear function w.r.t. Minkowski sums.

$\hookrightarrow V(sP_1 + tP_1', P_2, \ldots, P_d) = sV(P_1, P_2, \ldots, P_d) + t(P_1', P_2, \ldots, P_d)$

Exercise: Prove (A3).

Hint: "Shouldn't be too hard" using mixed subdivisions

Thm: Axioms (A1), (A2), (A3) above define mixed volume

How about all coeff's of $\mathrm{Vol}(t_1 P_1, \ldots, t_N P_N)$?

no longer require this be d

Prop: $\mathrm{Vol}(t_1P_1 + \dots + t_NP_N) = \sum_{\substack{k_1,\dots,k_N \geq 0 \\ \sum k_i = d}} \binom{d}{k_1 \dots k_N} V(\underbrace{P_1,\dots,P_1}_{k_1}, \dots, \underbrace{P_N,\dots,P_N}_{k_n}) t_1^{k_1} \cdots t_N^{k_n}$

Recall: Multinomial coeff $\binom{d}{k_1 \cdots k_n} = \frac{d!}{k_1!k_2!\cdots k_N!}$

$$= \sum_{(i_1,\dots,i_d)\in[N]^d} V(P_{i_1}, P_{i_2}, \dots, P_{i_N})\, t_{i_1} t_{i_2} \cdots t_{i_N} \quad (*)$$

(Top just combines like monomials from the bottom)

Proof: $\mathrm{Vol}(t_1P_1 + \dots + t_NP_N)$

$$\overset{(A1)}{=} \underbrace{V(t_1P_1 + \dots + t_NP_N, \dots, t_1P_1 + \dots + t_NP_N)}_{d \text{ times}}$$

which by (A2) and (A3) $= (*)$ ■

Bernstein - (Kushnirenko-Khovanskii) Thrm:

$A_i \subset \mathbb{Z}^d$ finite subsets for $i = 1, \dots, d$

Laurent polynomials $\quad x^a := x_1^{a_1} \cdots x_d^{a_d}$

$f_i(x_1,\dots,x_d) = \sum_{a\in A_i} c_{i,a} x^a$ for some constants $c_{i,a}$

$P_i = \mathrm{conv}(A_i) = \mathrm{Newton}(f_i)$ if $c_{i,a} \neq 0$

Thrm: For generic values of coeffs $c_{i,a}$, # distinct sltns in $(\mathbb{C}\setminus\{0\})^d$ of the system

$$(**) \quad \begin{cases} f_1(x_1,\dots,x_d) = 0 \\ f_d(x_1,\dots,x_d) = 0 \end{cases}$$

equals $d!\,\mathrm{Vol}(P_1,\dots,P_d)$

More precisely, $\exists$ algebraic subvariety (discriminental variety) $\mathcal{D} \subset \mathbb{C}^M$, $M = |A_1| + \dots + |A_d|$ s.t. $\forall (c_{i,a}) \in \mathbb{C}^M \setminus \mathcal{D}$, # sltns of $(**) = d!\,V(P_1,\dots,P_d)$

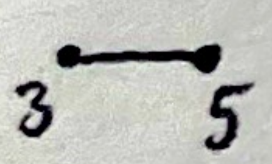
$\in \mathbb{Z}$ if P_i's are integer.

Ex.3) $d=1$

$f(x) = ax^3 + bx^4 + cx^5 = 0$

roots: ~~0, 0, 0~~, r_1, r_2 Thrm counts distinct non-zero roots ?

r_1, r_2 disinct if $(a,b,c) \notin \{(b^2-4ac)ac = 0\} \Rightarrow$ Then should get 2 roots

$P_1 =$  $\quad \mathrm{Vol}(P_1) = 2$

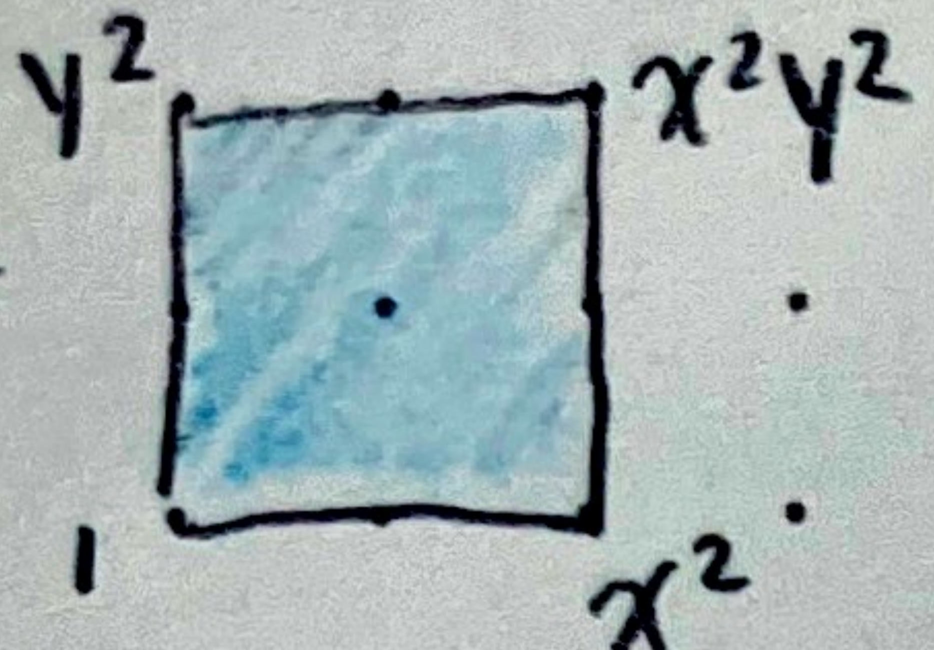
For $d=1$, Thrm $\longleftrightarrow$ Fundamental Thrm of algebra

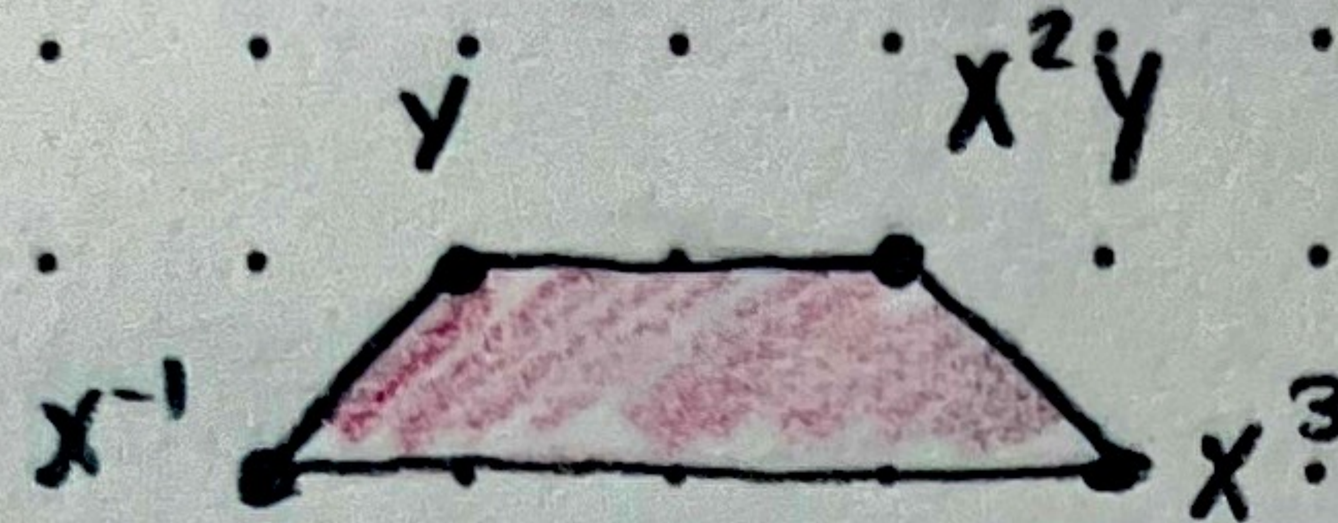
For $d=2$, two generic polys of degree m & n

$$\begin{cases} f(x,y)=0 \\ g(x,y)=0 \end{cases} \Longrightarrow \quad m\cdot n \text{ sols.} \Rightarrow \text{Bezout's Thrm}$$

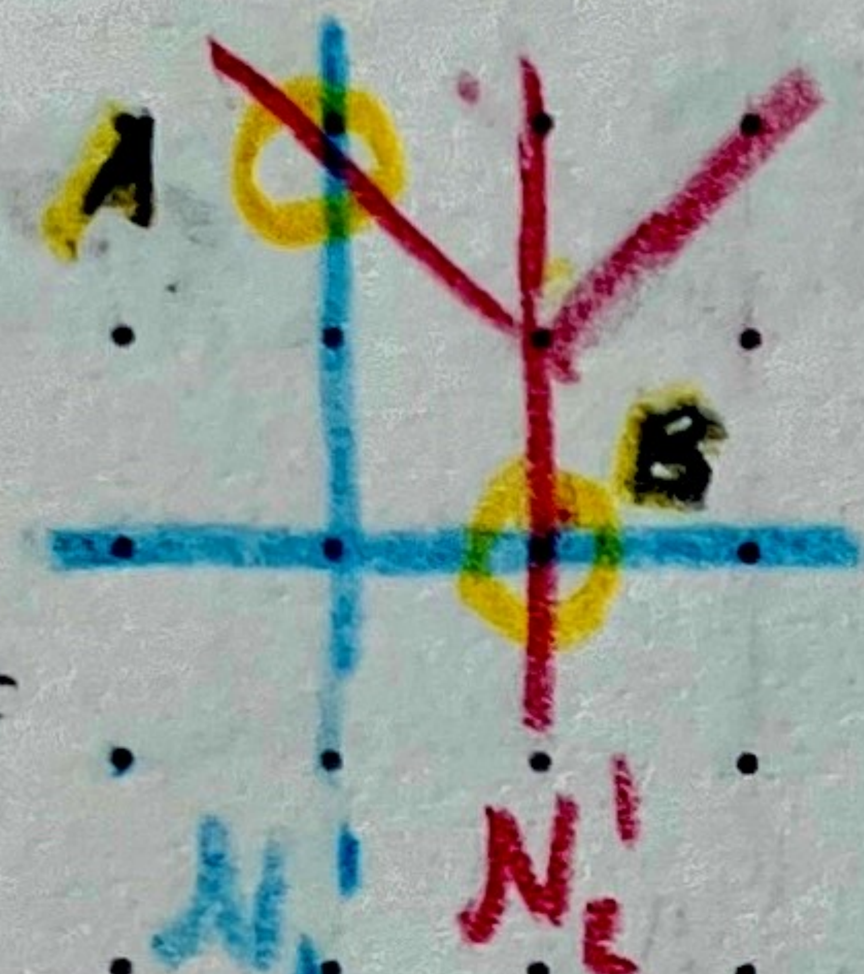
Ex. 4: $\begin{cases} a+bx^2+cy^2+dx^2y^2=0 \\ ex^{-1}+fx^3+gy+hx^2y \end{cases}$ $\quad a,b,\dots,h\in\mathbb{C}$ generic

$P_1 =$





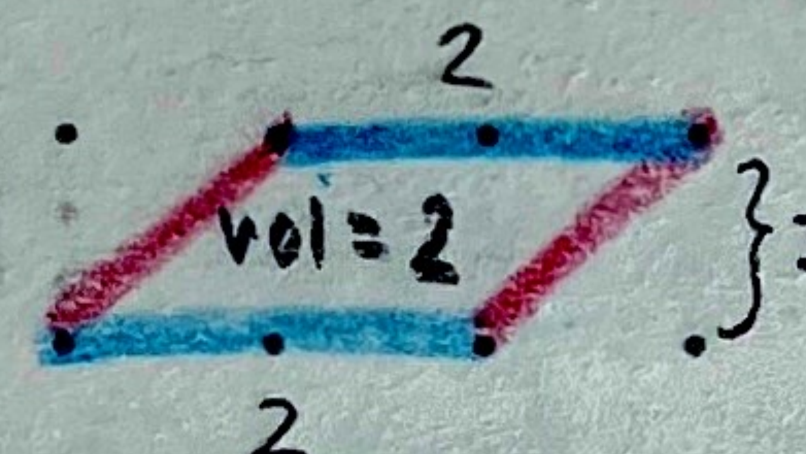
affine fan arrangement $\mathcal{A} =$



care about mixed points.

$\tilde{V}(P_1,P_2) = 2+8=10$

A:



B: