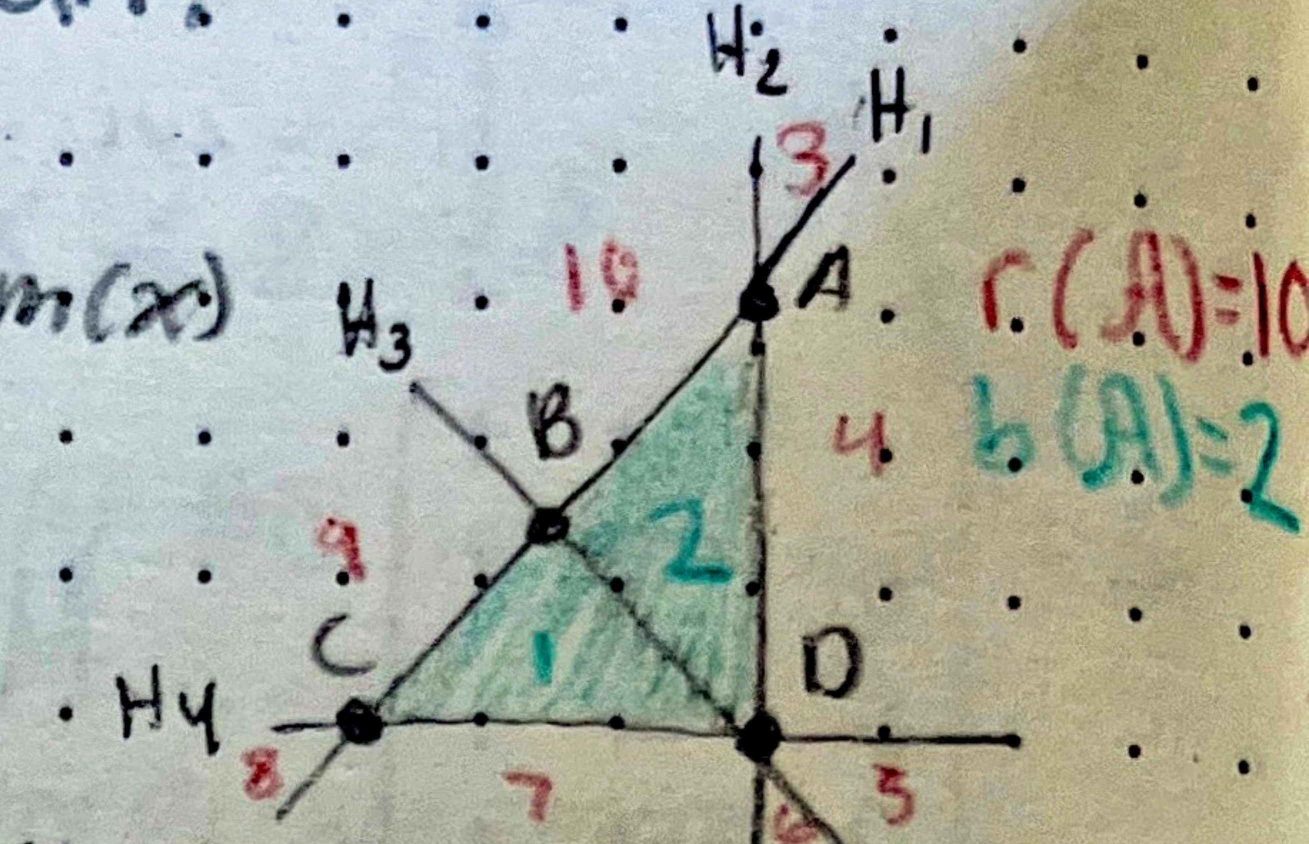
LECTURE 15 Wed 10/9

Last Time: $\mathcal{A} = \{H_1, \dots, H_N\}$ hyp. arr.
$L_{\mathcal{A}}$ its intersection (semi) lattice.

Char. poly $\chi_{\mathcal{A}}(t) := \sum_{x \in L_{\mathcal{A}}} \mu(\hat{0}, x)\, t^{\dim(x)}$

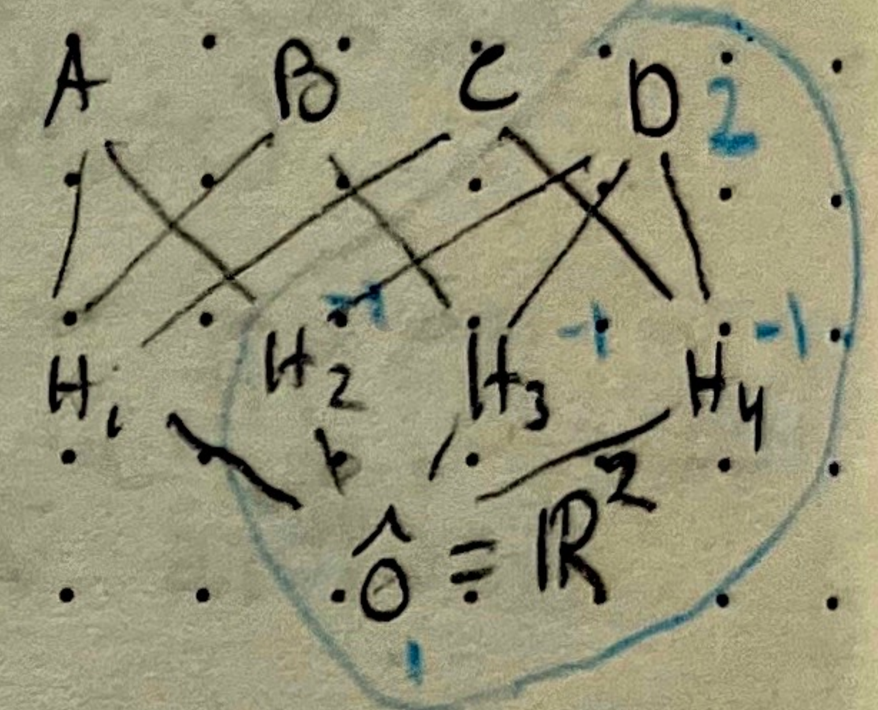
$r(\mathcal{A}) :=$ # regions of $\mathcal{A}$
$b(\mathcal{A}) :=$ # bounded regions of $\mathcal{A}$



Region bounded if it can be contained in sufficiently large ball

Thrm 1 (Zaslavsky's Thrm):
(1) $r(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1)$
(2) $b(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(1)$ if $\mathcal{A}$ is essential

Thrm 2: For a graphical arr. $\mathcal{A}_G$,
$\chi_{\mathcal{A}_G}(t) = \chi_G(t)$ (chromatic poly)



Proof by induction on N. First, we'll need some tools:
For graphs: Deletion contraction
$\chi_G(t) = \chi_{G \setminus e}(t) - \chi_{G/e}(t)$

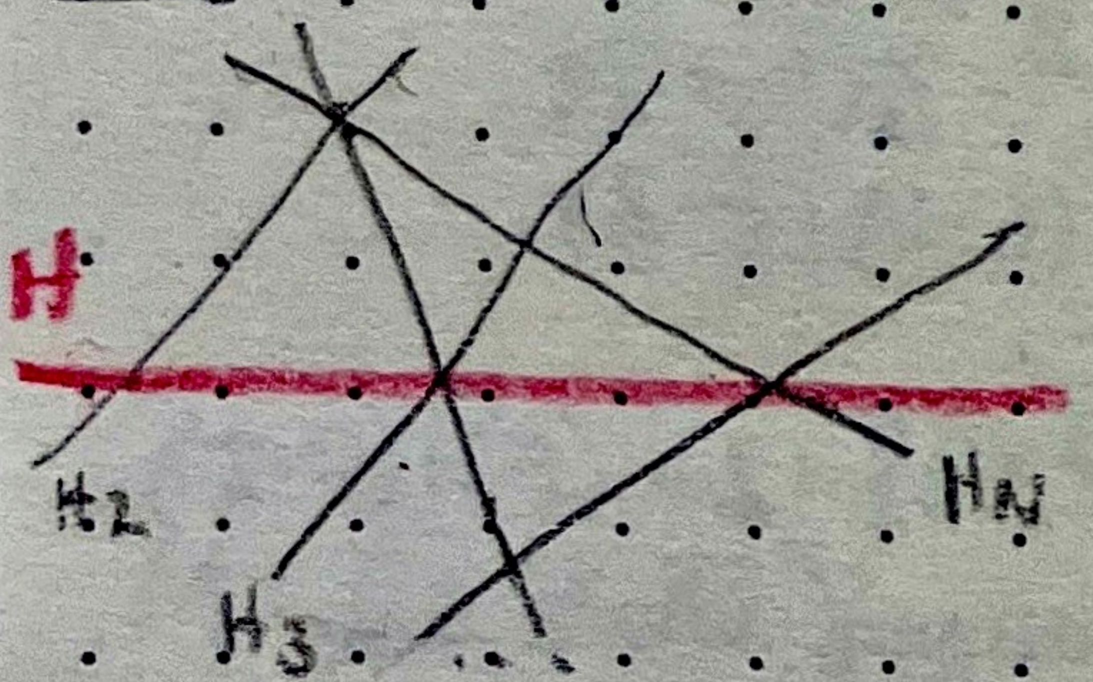
Deletion Restriction

Def: Let $H = H_1$. Assume $H \neq H_i$ $\forall\, i \geq 2$.
deletion: $\mathcal{A} \setminus H = \{H_2, \dots, H_N\}$

restriction: $\mathcal{A} \cap H = \{$all nonempty intersections $H \cap H_i\}$ a hyp. arr. in $\mathbb{R}^{n-1}$

Lemma 1: (1) $r(\mathcal{A}) = r(\mathcal{A} \setminus H) + r(\mathcal{A} \cap H)$
(2) $b(\mathcal{A}) = b(\mathcal{A} \setminus H) + b(\mathcal{A} \cap H)$ (if $\mathcal{A} \setminus H$ is essential)

Proof: (1) H cuts some regions of $\mathcal{A} \setminus H$ into 2 parts



number of such additional regions is exactly the # of regions in restriction

$\mathcal{A} \cap H$

$\Rightarrow r(\mathcal{A}) - r(\mathcal{A} \setminus H) = r(\mathcal{A} \cap H)$

(2) 3 cases: (I) H cuts unbounded region into 2 unbounded regions
(II) cut unbounded into 1 bounded & 1 unbounded
(III) cuts bounded region into 2 bounded regions

In all 3 cases, #. additional bounded regions

$b(\mathcal{A}) - b(\mathcal{A} \setminus H) = b(\mathcal{A} \cap H)$

(I). happens exactly when restriction to H of intersection w/. that region is not bnded

(II). happens when restriction w/ region has 1 bnded component

(III) restriction to region has 1 bnded component as well

Lemma 2: $\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A} \setminus H}(t) - \chi_{\mathcal{A} \cap H}(t)$

Now Thrms 1 & 2 follow by induction

Note: Base case different for $r(\mathcal{A})$ vs. $b(\mathcal{A})$

↑ empty arr. ← minimal essential arr.

Thrm (Whitney's Thrm): $\mathcal{A} = \{H_1, \dots, H_N\}$

$$\chi_{\mathcal{A}}(t) = \sum_{\substack{I \subseteq [N] \text{ s.t.} \\ \bigcap_{i \in I} H_i \neq \emptyset}} (-1)^{|I|} t^{\dim(\bigcap_{i \in I} H_i)}$$

Pretty similar to def: $\bigcap_{i \in I} H_i$ in intersection lattice, but could get same pt, for multiple I

Ex. For $\mathcal{A}$ from earlier ↙ pick all of H_1, H_2, H_3

$\mu(\hat{0}, D) = 3(-1)^2 + (-1)^3 = 2$

↑ pick any 2 of H_2, H_3, H_4

Proof Whitney's Thrm ⇒ Lemma 2:

$$\chi_{\mathcal{A}}(t) \underset{\text{Whitney's Thrm}}{=} \sum_{I \subseteq [N]} \cdots = \underbrace{\sum_{I \text{ s.t.}} \cdots}_{\mathcal{A} \setminus H \text{ part}} + \underbrace{\sum_{I \text{ s.t.}}}_{\mathcal{A} \cap H \text{ part}} \underset{\text{Whitneys Thrm again}}{=} \chi_{\mathcal{A} \setminus H}(t) + \chi_{\mathcal{A} \cap H}(t)$$

Need $\mu(\hat{0}, X) \overset{?}{=} \sum_{I \subseteq [N] \text{ s.t. } \bigcap_{i \in I} H_i = X} (-1)^{I}$

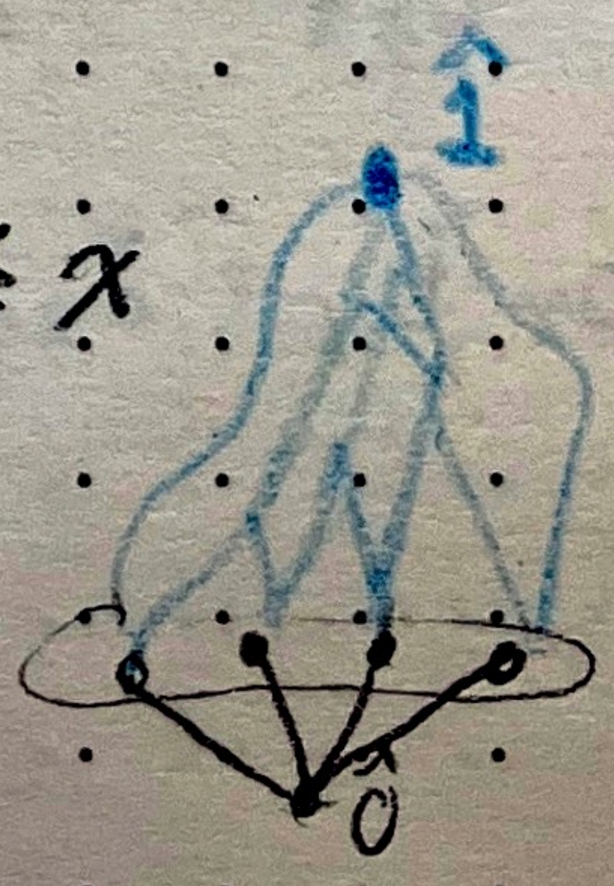
Rota's Crosscut Theorem

L any finite lattice

$C \subseteq L \setminus \{\hat{0}\}$ s.t. $\forall x \in L \setminus \{\hat{0}\} \ \exists c \in C$ s.t. $c \leq x$

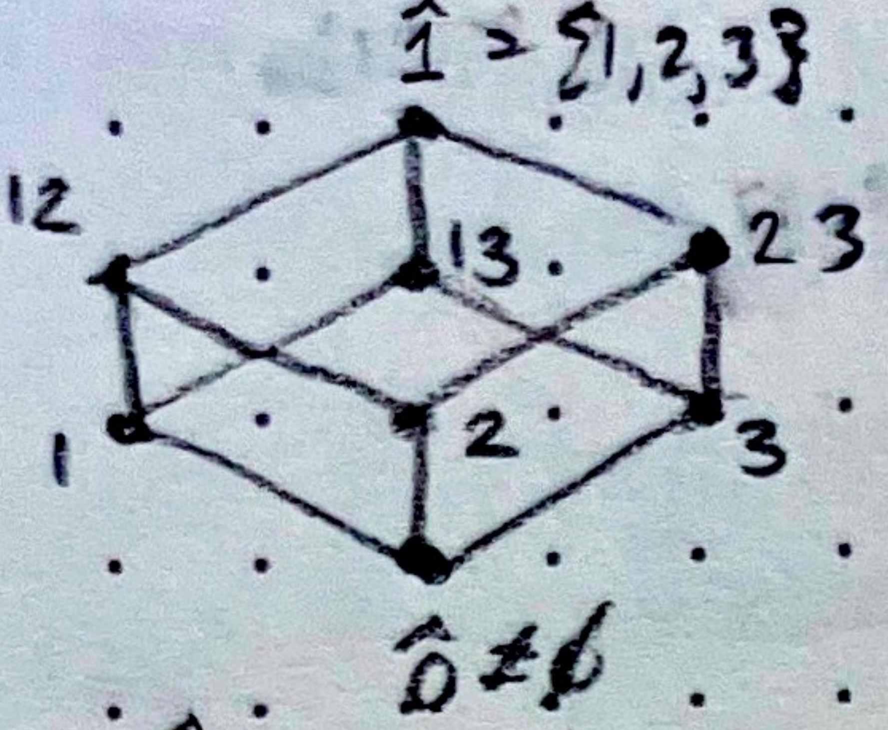
Such C is called a lower crosscut of L

Then $\mu_L(\hat{0}, \hat{1}) = \sum_{\substack{B \subseteq C \text{ s.t.} \\ \text{join } \bigvee_{b \in B} = \hat{1}}} (-1)^{B}$

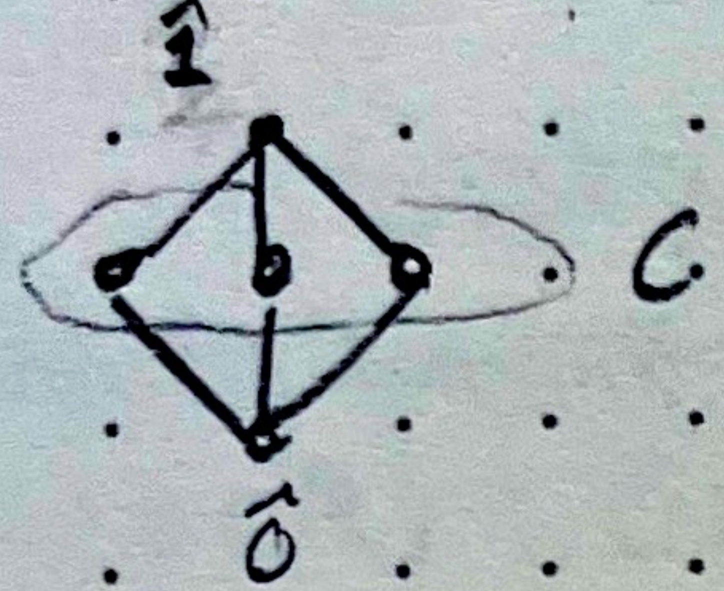
C 

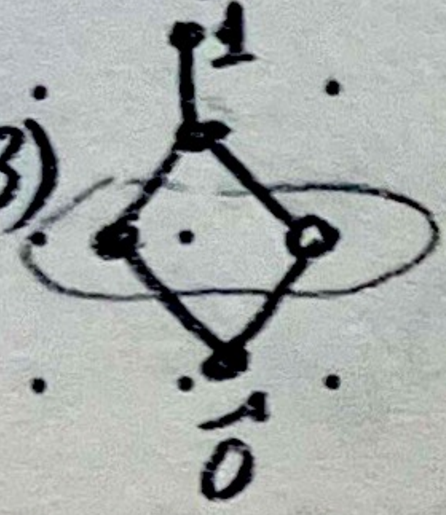
Ex. (1): Boolean lattice.

B_n = {All subsets of $[n]$ ordered by inclusion}

$B_3 =$ 

$\mu_{B_n}(\hat{0}, \hat{1}) = (-1)^n$

Ex. (2)  $\mu(\hat{0}, \hat{1}) = 3(-1)^2 + (-1)^3 = 2$

Ex. (3)  $\mu(\hat{0}, \hat{1}) = 0$

To prove Crosscut Thrm, we'll need

Def: Möbius Algebra $A(L)$ of a finite lattice

$A(L) \simeq \mathbb{R}^{|L|}$ (as a vector space)

$\mathbb{R}$-linear basis a_x, $x \in L$

multiplication $a_x \cdot a_y = a_{x \vee y}$

Commutative, associative algebra. Identity $\mathbb{1} = a_{\hat{0}}$

Another linear basis:

$(*)$ $\quad b_x = \sum_{y \geq x} \mu(x, y)\, a_y \quad \forall x \in L$

$\Updownarrow$

$(**)$ $\quad a_x = \sum_{y \geq x} b_y$

Lemma: $b_x \cdot b_y = \delta_{xy} \cdot b_x$ where $\delta_{xy} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$