LECTURE 17 Wed 10/16

Finite Field Method

Fix $\mathcal{A} = \{H_1, \ldots, H_N\}$ hyp. arr. in $\mathbb{R}^n$.

$H_i = \{\vec{x} \in \mathbb{R}^n \mid (\vec{v}_i, \vec{x}) = h_i\}$

Assume all coeffs in $\mathbb{Z}$.

$\mathbb{F}_q$ finite field w/ $q = p^r$ elts (p prime).

$\mathcal{A}_q = \{H_1^q, \ldots, H_N^q\}$ $\quad H_i^q \subset \mathbb{F}_q^n$

H_i^q given by the same eqns but now over $\mathbb{F}_q$

Def: $\mathcal{A}$ has a good reduction over $\mathbb{F}_q$ is $\mathcal{A}_q$ is a hyperplane arr. & its intersection semmilattice $L_{\mathcal{A}_q} \simeq L_{\mathcal{A}}$.

was Lemma in last lecture, now promoted to thrm.

Thrm1: $\exists$ a finite set P_{bad} of primes s.t. $\forall q = p^r$, $p \notin P_{bad}$ $\mathcal{A}$ has good reduction over $\mathbb{F}_q$.

Thrm2: If $\mathcal{A}$ has a good reduction $/\mathbb{F}_q$, then

$$\boxed{\chi_{\mathcal{A}}(q) = \#\left(\mathbb{F}_q^n \setminus \bigcup_{H \in \mathcal{A}_q} H\right)}$$

Proof of Thrm 1: What exactly are the set of primes where things can go wrong:

Define 2 matrices $A = [\vec{v}_1, \ldots, \vec{v}_N]$ $n \times N$ matrix.

$$\tilde{A} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_N \\ h_1 & h_2 & \cdots & h_N \end{bmatrix} \quad (n+1) \times N \text{ matrix}$$

Things can go wrong exactly when maximal minors are zero.

$\Rightarrow P_{bad} = \{$all prime factors of all non-zero max minors of A and $\tilde{A}\}$

$\hookrightarrow$ More on why this is true on next page.

To do this will need

Matroid data of A

basis: n-elt set $B \subseteq [N]$ s.t. $\det(\vec{v}_i \mid i \in B) \neq 0$

indep sets $I \subseteq [N]$ s.t. $\exists$ basis B s.t. $B \supseteq I$

Will define matroids in full generality later, but for now can take these as def for this special case

rank function $\forall J \subseteq [N]$
$$\mathrm{rank}_A(J) = \max_{\substack{I \subseteq J \\ I \text{ indep.}}} |I|$$

flats: $F \subseteq [N]$ is a flat if $\forall i \in [N] \setminus F$, $\mathrm{rank}(F \cup \{i\}) \gneq \mathrm{rank}\, F$

lattice of flats: All flats ordered by inclusion

Lemma 1: If A is central arr. (all $h_i = 0$) then its intersection lattice $L_A \simeq$ lattice of flats

Lemma 2: $\bigcap_{i \in I} H_i \neq \emptyset \iff$ For any lin. dep. between $\vec{v}_i$, $i \in I$ we have same lin. dep. between $\binom{\vec{v}_i}{h_i}$

$$\iff \mathrm{rank}_A(I) = \mathrm{rank}_{\tilde{A}}(I)$$

Proof of Thm 2:

$$\chi_A(q) = \chi_{A_q}(q) \overset{\text{claim}}{=} RHS$$

Prove claim by induction on N

base: $N = 0 \rightarrow$ $q^n = q^n = \#$ pts in $\mathbb{F}_q^n$

Step: χ_{A_q} satisfies del-restriction recurrence.

"RHS obviously satisfies deletion restriction"

$$\#\left(\mathbb{F}_q^n \setminus \bigcup_{i=1}^{N} H_i^q\right) = \#\left(\mathbb{F}_q^n \setminus \bigcup_{i=2}^{n} H_i^q\right) - \#\left(H_1^q \setminus \bigcup_{i=2}^{n} (H_i^q \cap H_1^q)\right)$$

How to use this?

Graphical Array A_G

$$\chi_{A_G} = \#\left((x_1, \ldots, x_n) \in \mathbb{F}_q^n \mid x_i \neq x_j \ \forall \text{ edge } (i,j) \text{ of } G\right)$$

Cor: $\chi_{A_G}(q) = \chi_G(q)$

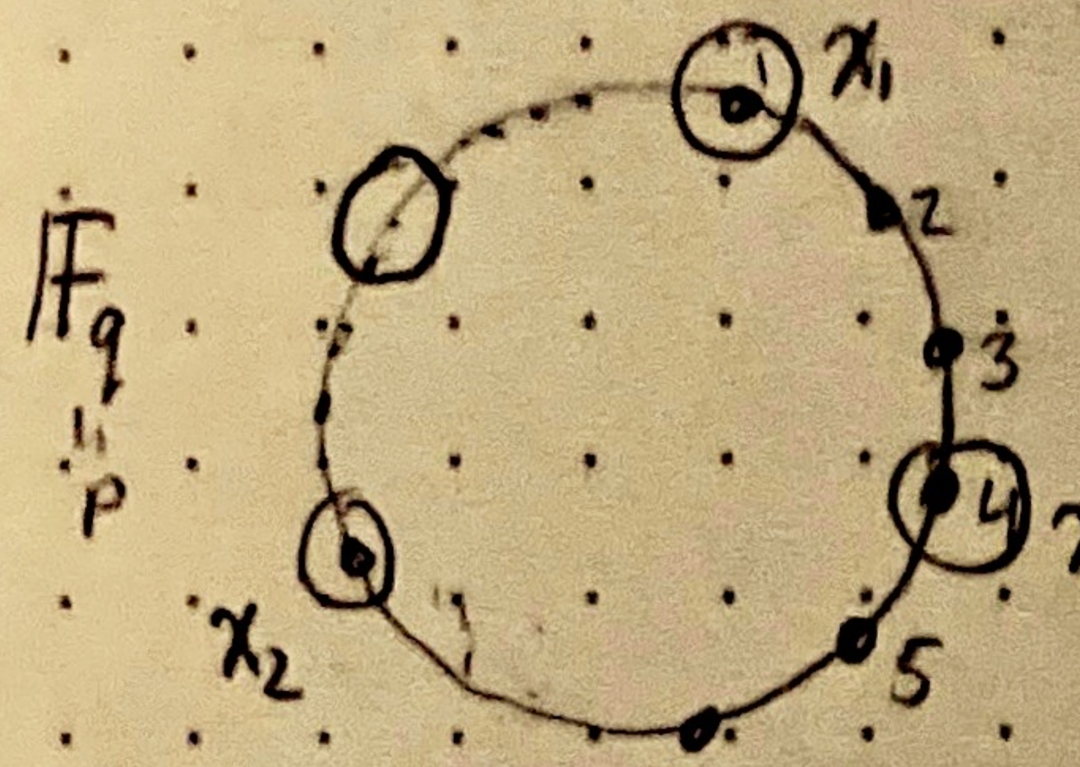
$$\chi_{A_{K_n}}(q) = q(q-1)(q-2)\cdots(q-n+1)$$

Catalan Arrangement

Cat_n: $x_i - x_j = -1, 0, 1$ $\forall\, 1 \le i < j \le n$

$$\chi_{Cat_n}(q) = \#\{(x_1, \dots, x_n) \in \mathbb{F}_q^n \mid x_i - x_j \neq -1, 0, 1\}$$

Assume $q = p$ where p is a sufficiently large prime



place n laballed balls on some spots on the circle

$\chi_{Cat_n}(q)$ same as requiring no balls in same or adjacent positions.

How many ways can we do this?

By symmetry, assume x_1 goes in spot 1 (factor of q)
Assume #'s go in order (factor of $(n-1)!$)

$\Rightarrow \chi_{Cat_n}(q) = q\,(n-1)! \cdot \#\{$ways to place $n-1$ balls on $2, 3, \dots, q$ s.t. balls are ≥ 2 apart$\}$

same as $\#\{c_1 + \dots + c_{n-1} = q$ s.t. $c_i \geq 2\}$ $\leftarrow$ c_i's distance between balls

Can get regular choose function by subtracting 1 space after each ball

$\hookrightarrow = \chi_{Cat_n}(q) = q(n-1)!\binom{q-n-1}{n-1}$

<u>Thrm</u>: $\chi_{Cat_n}(q) = q(q-n-1)(q-n-2)\cdots(q-2n+1)$ (n factors)

regions $|\chi_{Cat_n}(-1)| = n!\,\frac{1}{n+1}\binom{2n}{n}$

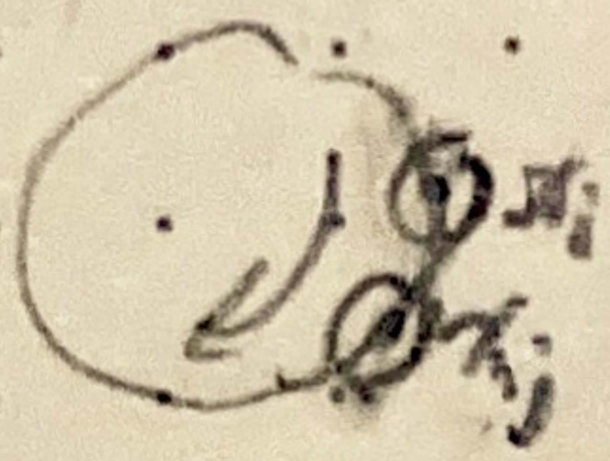
<u>Shi arrangement</u>

Shi_n: $x_i - x_j = 0, 1$ $1 \le i < j \le n$

Assume $q = p$. (p a sufficiently large prime)

$$\chi_{Shi_n}(q) = \#\{(x_1, \dots, x_n) \in \mathbb{F}_q^n \mid x_i - x_j \neq 0, 1 \text{ for } i < j\}$$

similar ball placement but we allow balls in adjacent positions

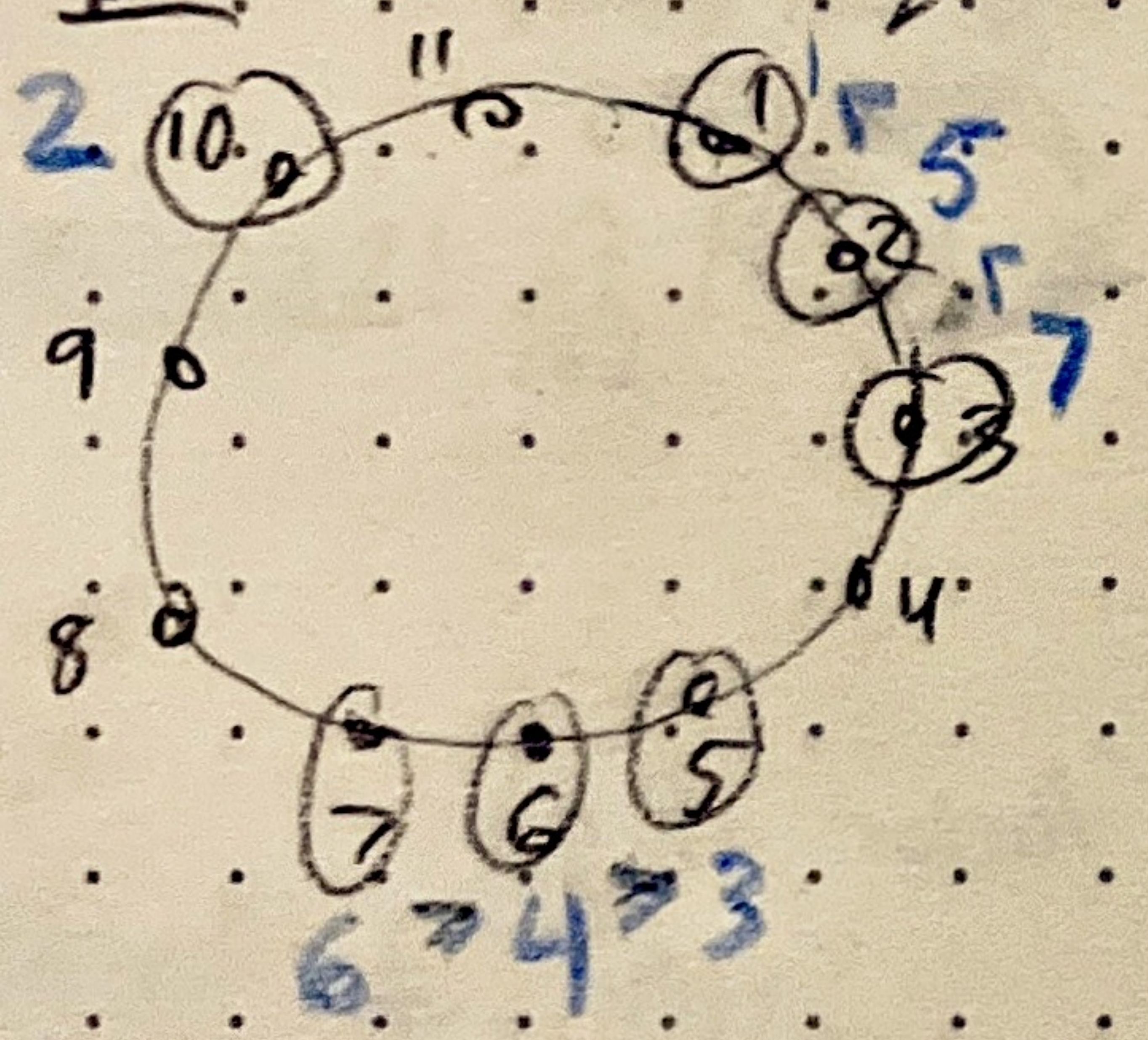


only if $i < j$

Assume $x_1 = 1$.

such ball placements $\overset{\text{bij}}{\longleftrightarrow}$ weak ordered set partitions $\pi = (B_1 | B_2 | \dots | B_{q-n})$ of $[n]$ with $q-n$ (possibly empty blocks) and $1 \in B_1$.

<u>E.x.</u> $n = 7$ $q = 11$



$\rightsquigarrow$ $\pi = (157 | 346 | \emptyset | 2)$

$\# \pi\text{'s} = (q-n)^{n-1}$