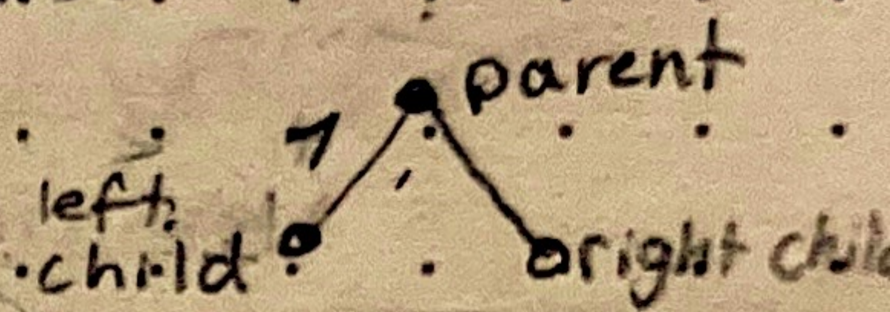
LECTURE 19 Mon 10/21

Last Weak: Braid, Catalan, Shi, Liniel arrangements
hyperplanes $x_i - x_j = k$ $1 \le i < j \le n$

Thrm: # regions of ↑ = # (special kind of) labelled binary tree on n verts

(1) Catalan: $n!C_n$ = # all labelled binary trees
(2) Shi: $(n+1)^{n-1}$ = # left increasing binary trees



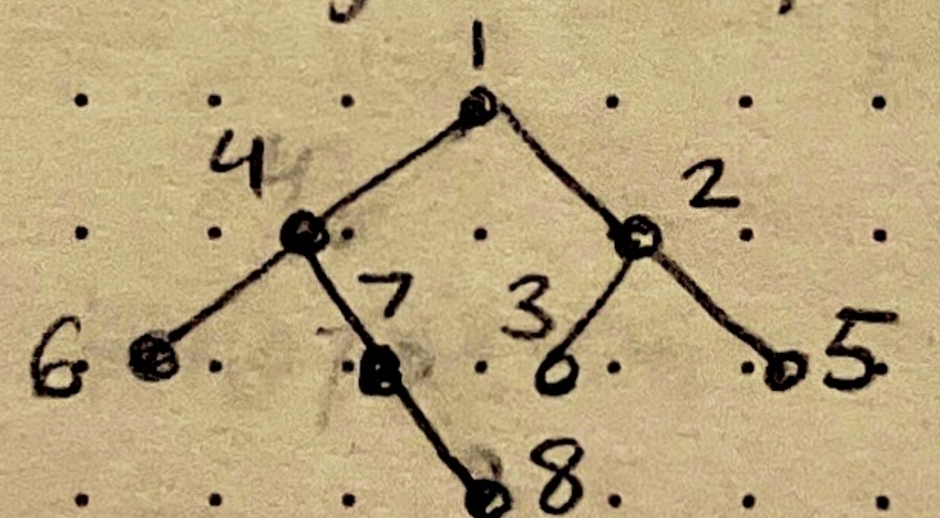
(3) Liniel: $\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(k+1)^{n-1}$ = #LBS trees

(4) Braid arr: $n!$ = # increasing binary trees

Bijections:

(4) {permutations} ⟷ {increasing binary trees}

6 4 7 8 1 3 2 5



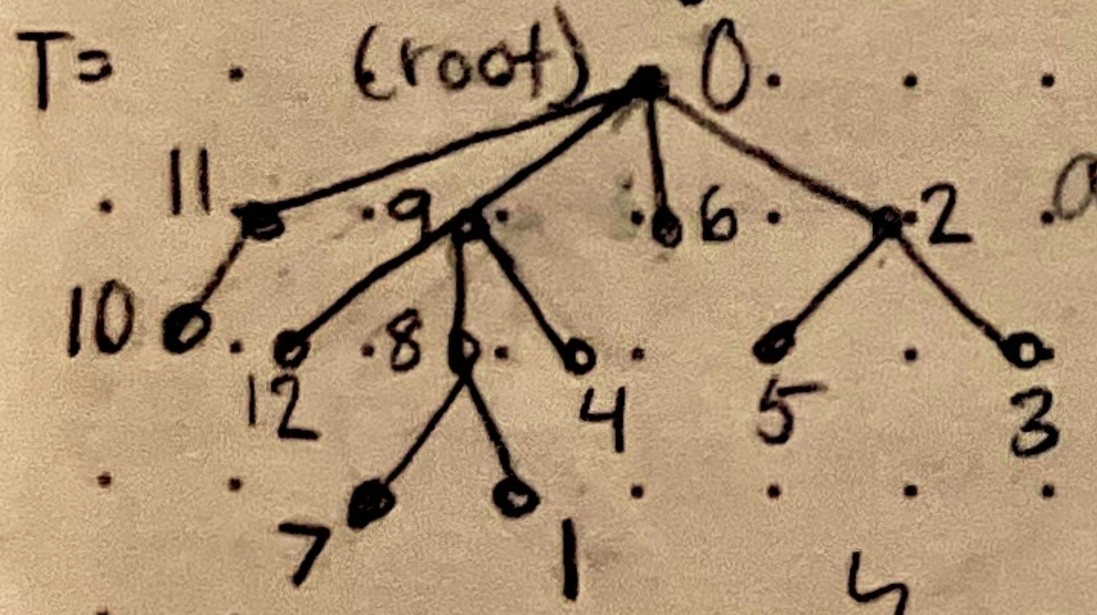
lowest entry is root
left (right) side becomes
left (right) tree off of root, keep applying same procedure recursively.

(3) {all labelled trees on n+1 verts} ⟷ {left increasing binary trees on n vertices}
$(n+1)^{n-1}$

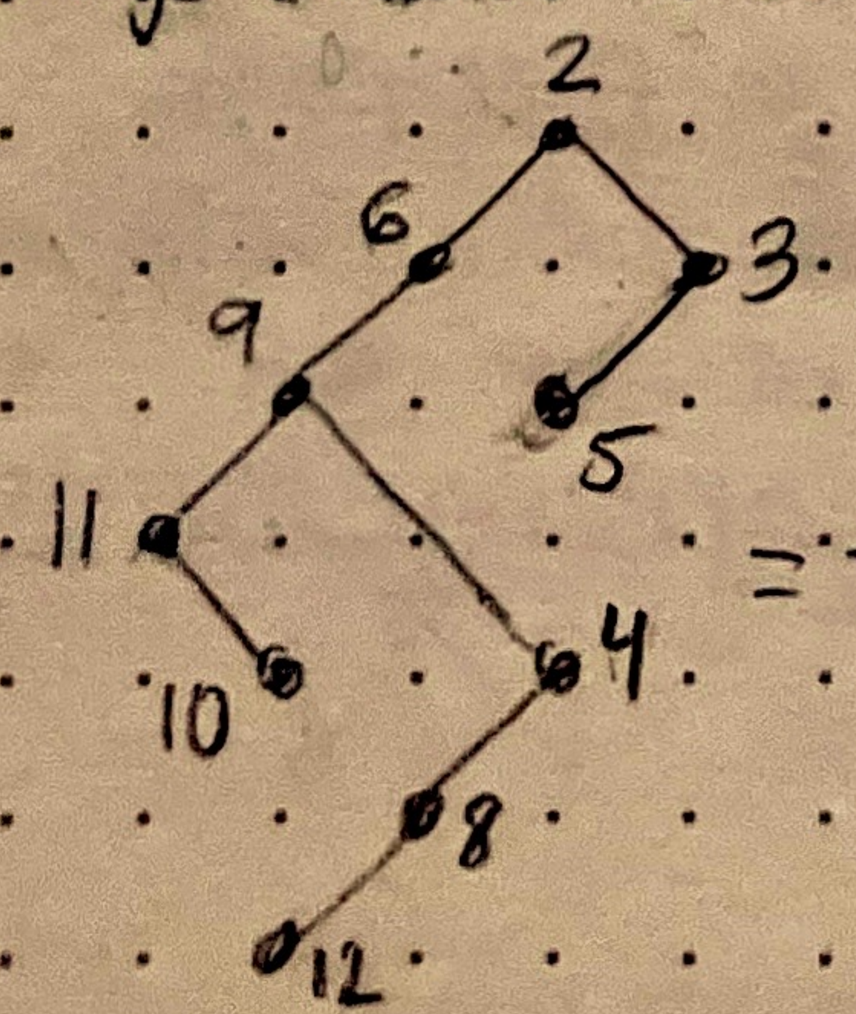
T ⟶ T'

Ex. Starting w/ abstract tree, draw in plane in specific way:

T=

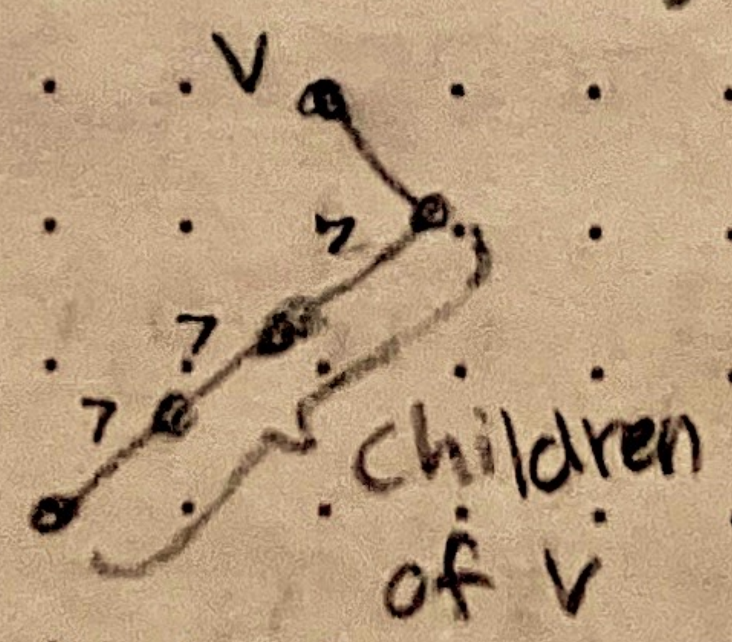


arrange labels left to right in decr. order.

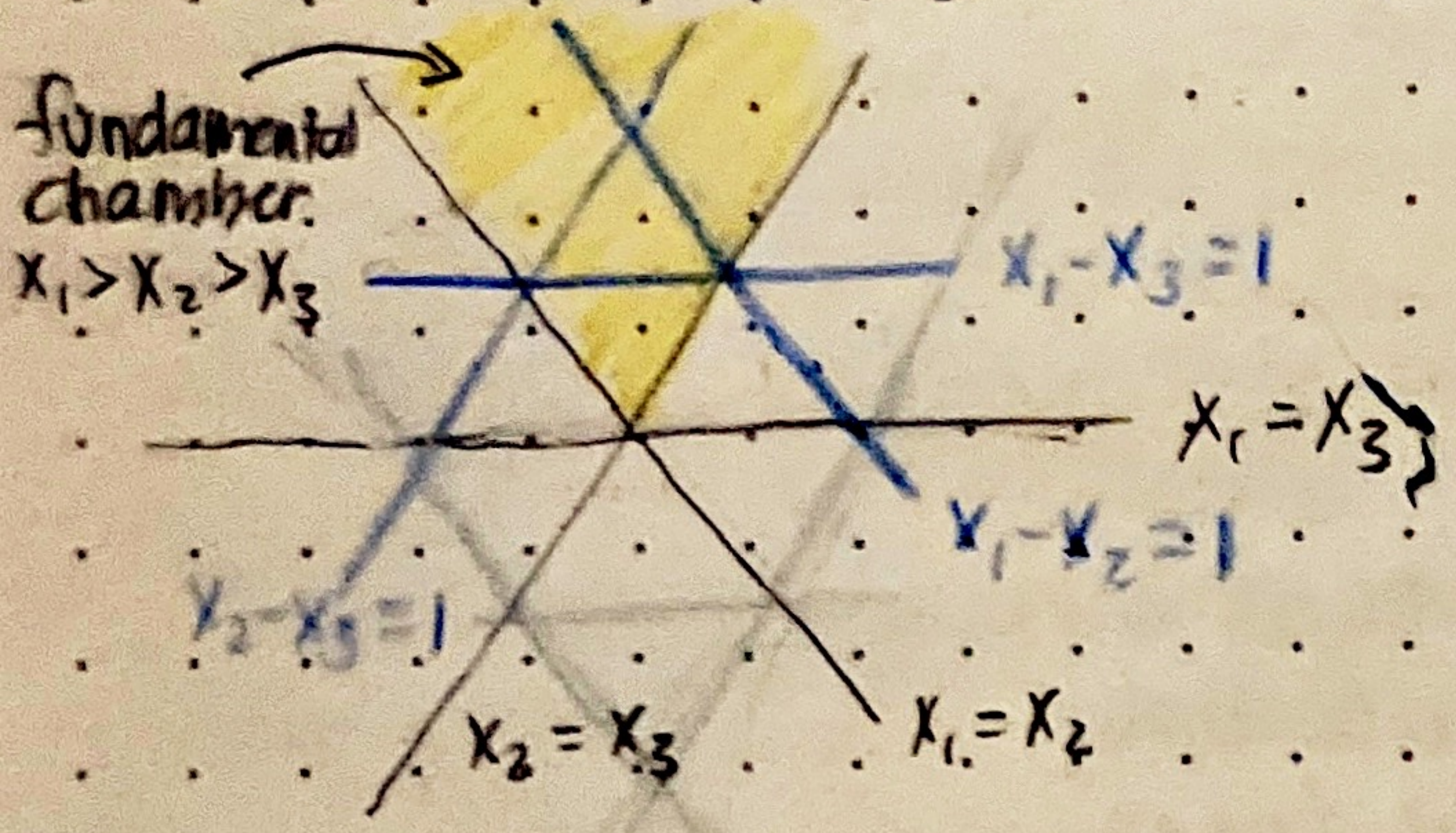


= T'

Given vertex v, all its children are in chain going down left.



(1) $\left\{\begin{array}{l}\text{regions of Catalan arr.} \\ \text{inside the fundamental} \\ \text{chamber } x_1 > x_2 > \dots > x_n\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Dyck paths} \\ \text{w/ } 2n \text{ steps}\end{array}\right\}$



$x_i - x_j > 0 \quad i < j$

$x_i - x_j \overset{?}{<} \text{ or } \overset{?}{>} 1$

R regions inside of the fund. chamber

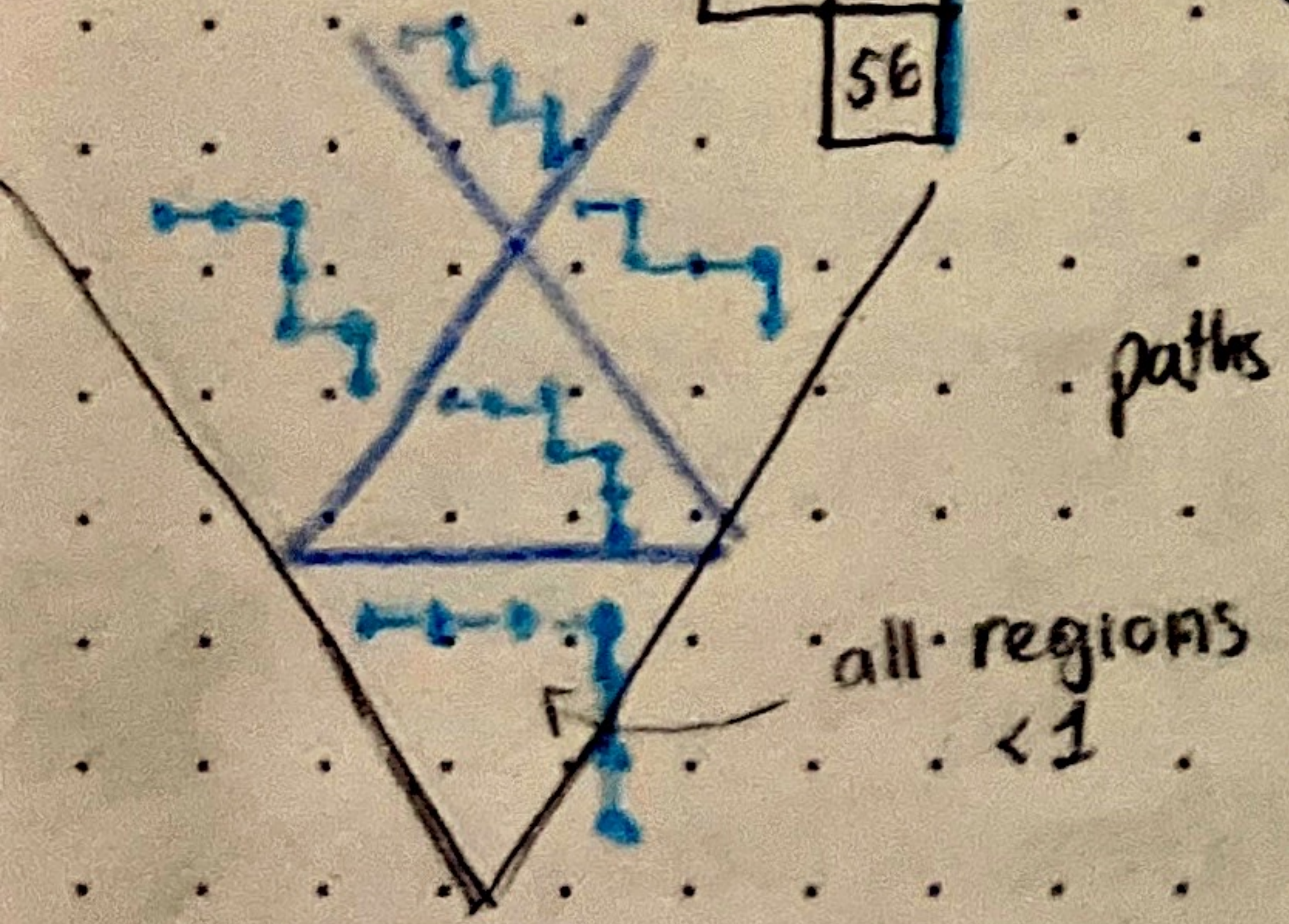
$S = \{(i,j) \mid \begin{array}{l} 1 \le i < j \le n \\ x_i - x_j < 1 \end{array}\}$

Arrange all pairs (i,j) in stairase shape

12	13	14	15	16
	23	24	25	26
		34	35	36
			45	46
				56

If $(i,j) \in S$, then $(i',j') \in S$ for any $i \le i' < j' \le j$.

$\Rightarrow$ S corresponds to the set of boxes below a Dyck path



paths on grid

12 13 14
23 24
34

Clear this is injective.

Proof of surjectivity by cheating:
We know # regions is C_n and so is # paths, so must be surjection.

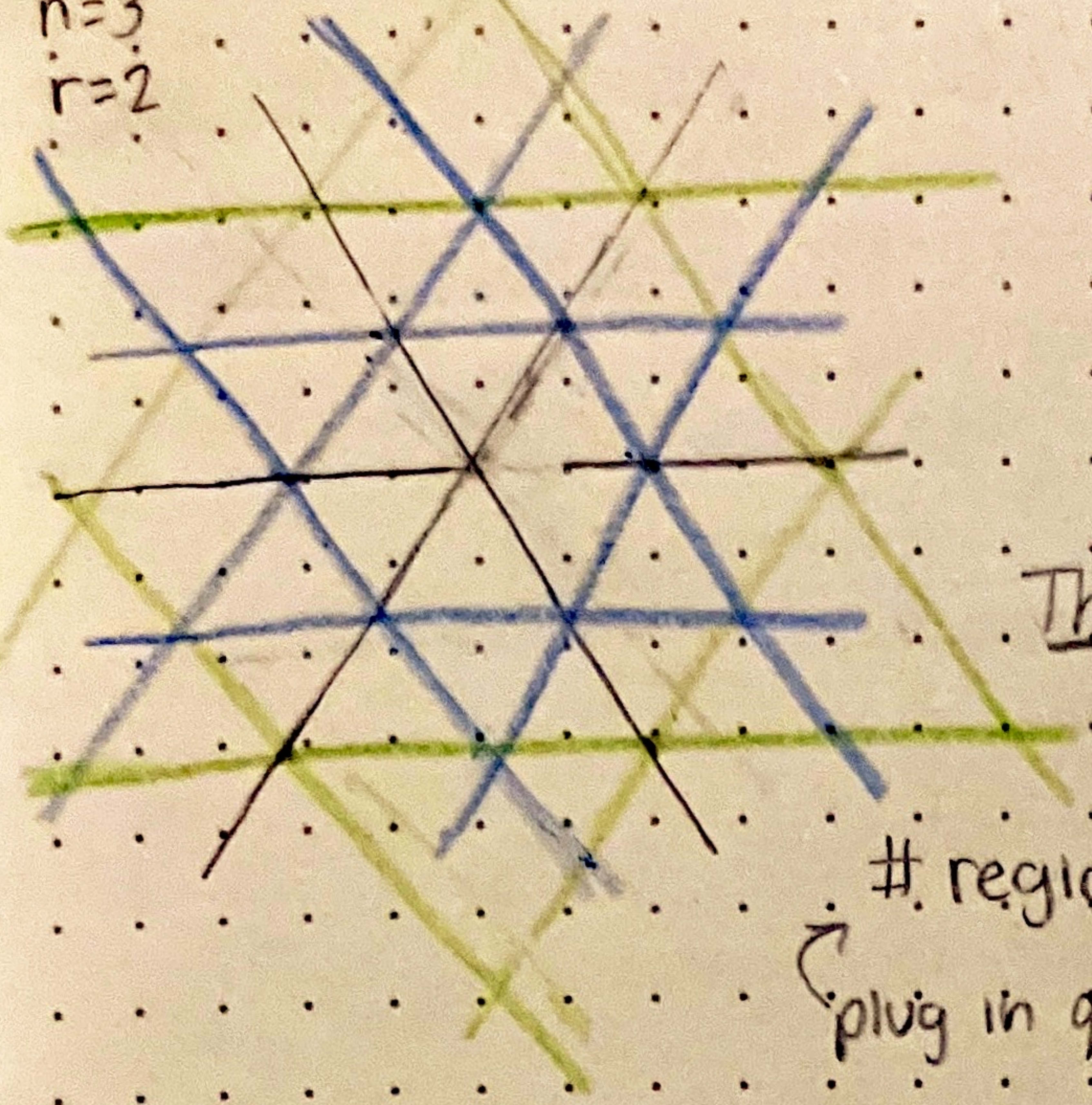
Alternatively, can define partial order on regions, and this poset is isomorphic to poset on Dyck paths.
(go up by collapsing a box so new path contained inside old path)

Extended Catalan & Shi arrangements

Ext. Catalan: Fix n, r

hyperplanes $x_i - x_j = -r, -r+1, \dots, r \qquad 1 \le i < j \le n$

n=3
r=2



Char. poly.

$\tilde{\chi}_{\mathcal{A}}(q) := \frac{1}{q}\chi_{\mathcal{A}}(q)$

a poly. of deg $n-1$

<u>Thrm</u>: $\tilde{\chi}_{ext.\,Cat.}(q) = (q-rn-1)(q-rn-2)\cdots(q-rn-n+1)$

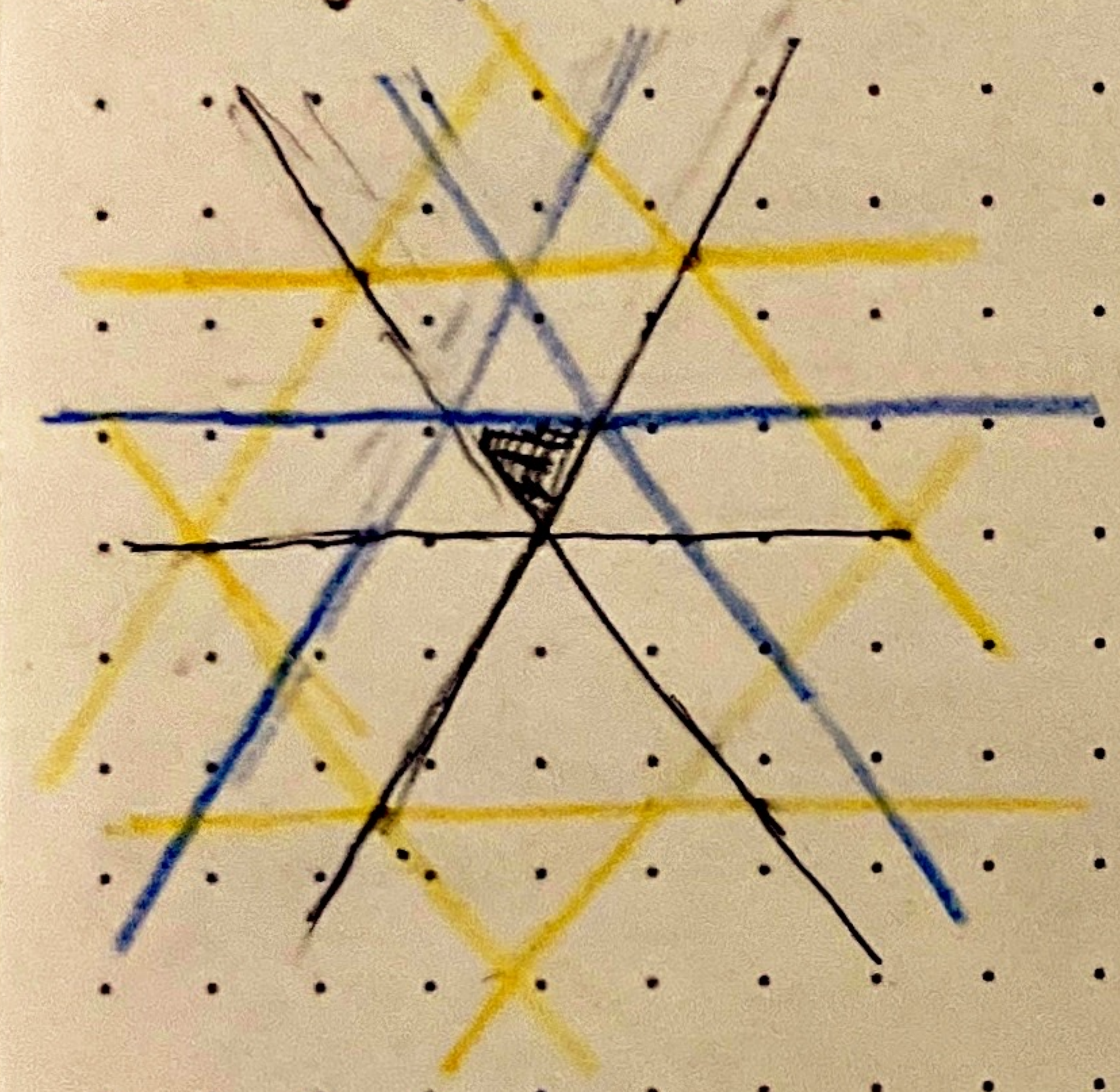
$\#$ regions $= n!\boxed{\frac{1}{rn+1}\binom{(r+1)n}{n}}$ ← extended Catalan #.

plug in $q=-1$

A special case of Fuss-Catalan #

<u>Extended Shi arrangement</u>:

Fix n, r. Hyperplanes $x_i - x_j = -r+1, -r+2, \ldots, r$



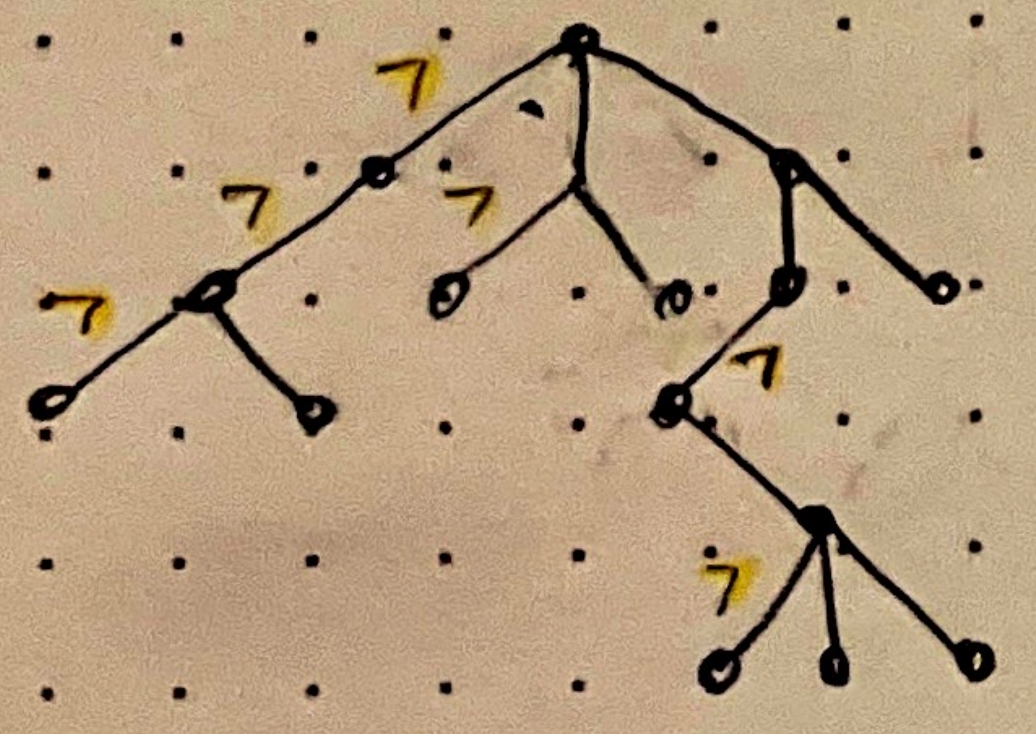
<u>Thrm</u>: $\tilde{\chi}(q) = (q-n\cdot r)^{n-1}$

$\#$ regions $= \boxed{(nr+1)^{n-1}}$ ← extended Cayley #'s.

Instead of binary trees, these are counted by $(r+1)$-ary trees

(each node has up to $r+1$ children, we keep track of which of the $r+1$ slots each child is in)

Ex. $r=2$
trinary tree.



$\#$ $(r+1)$-ary trees on n nodes is $n!\frac{1}{rn+1}\binom{(r+1)n}{n}$

$\#$ left-incr. $(r+1)$-ary trees on n nodes is $(nr+1)^{n-1}$.

just the very leftmost of the $r+1$ spots has to be increasing