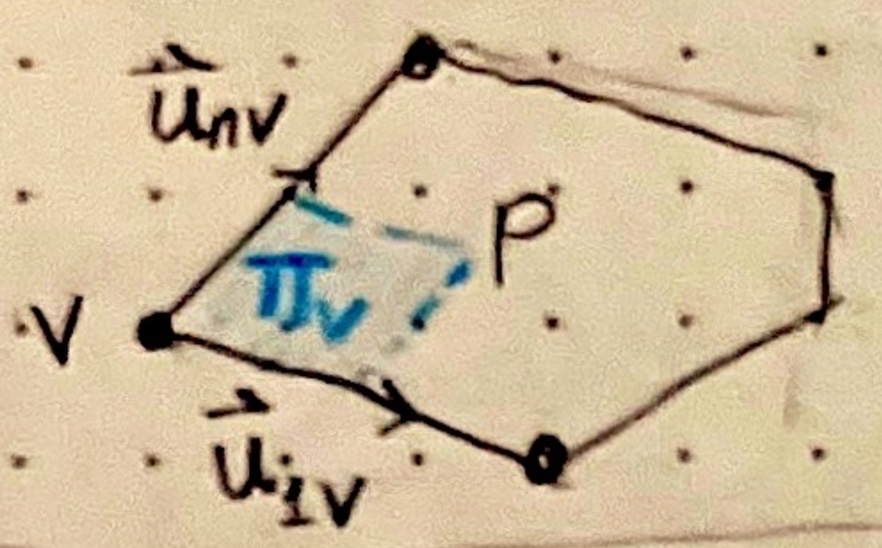
LECTURE 26 Wed 11/6

Brion: $P \subset \mathbb{R}^n$ simple lattice polytope



$\vec{u}_{1v}, \ldots, \vec{u}_{nv} \in \mathbb{Z}^n$ vector generating edges out of vertex v

$$(*) \quad \sum_{a \in P \cap \mathbb{Z}^n} x^a = \sum_{v \text{ vertex of } P} \frac{\sum_{b \in \pi_v \cap \mathbb{Z}^n} x^b}{\prod_{i=1}^{n} (1 - x^{\vec{u}_{iv}})}$$

π_v = fundamental half open parallelotope

Thrm 1: $\exists!$ map $S: A \longrightarrow \mathbb{R}(x_1, \ldots, x_n)$

space of rational polyhedra spanned by $[P]$ in $\mathbb{R}^n$

The field of rat. functions

s.t. (1) $S([\circ]) = 1$

the pt. 0

(2) $S([P+a]) = x^a S([P]) \quad \forall P \ \& \ a \in \mathbb{Z}^n$

$A' \subseteq A$ the subspace spanned by ruled polyhedra (polyhedra w/out vertices)

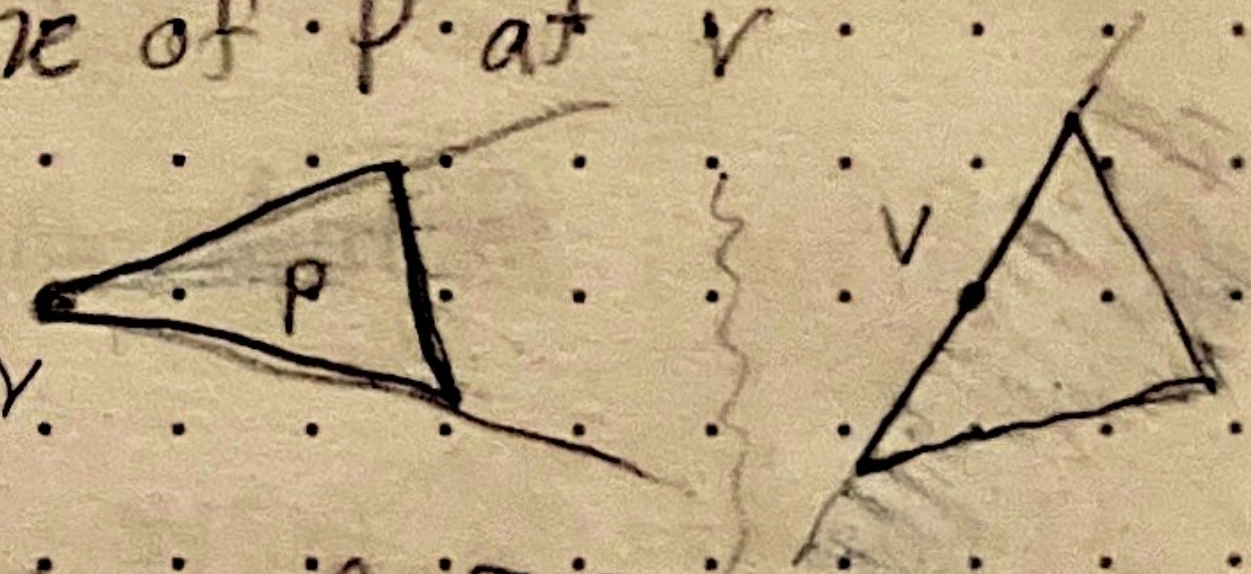
In posted notes from 3 yrs. ago, these are called non-essential polyhedra

Thrm 2: In A/A'

$$[P] \equiv \sum_{v \text{ vertex of } P} [C_{v,p}]$$

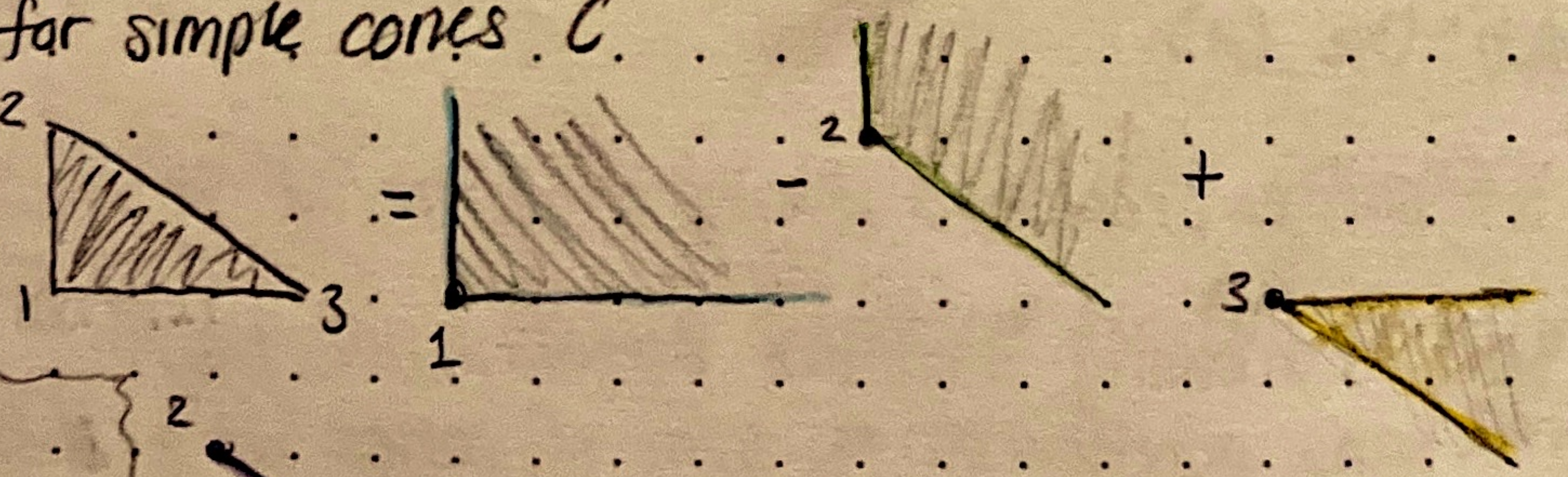
local cone of P at v

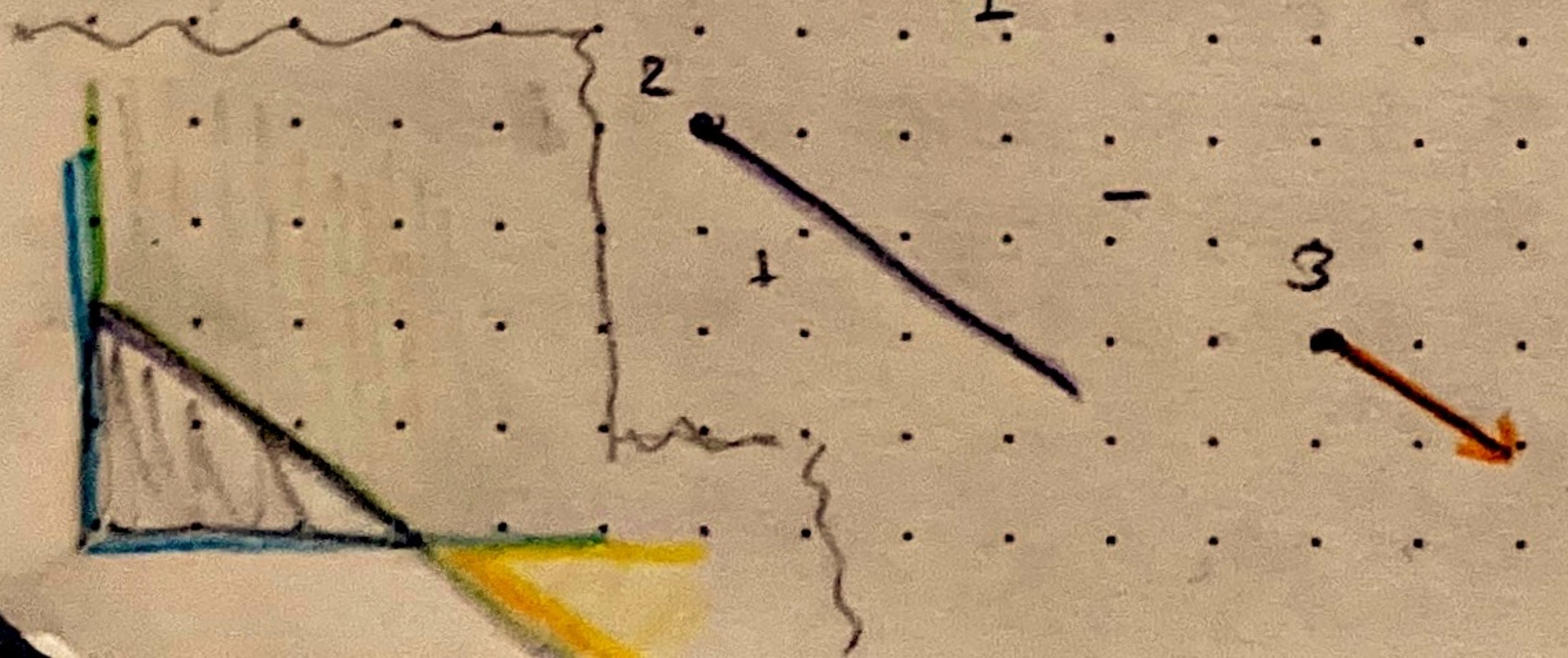
$C_{v,p}$ =



Lemma: Any elt of A is a lin. combination of $[C]$ for simple cones C.

Ex. P =

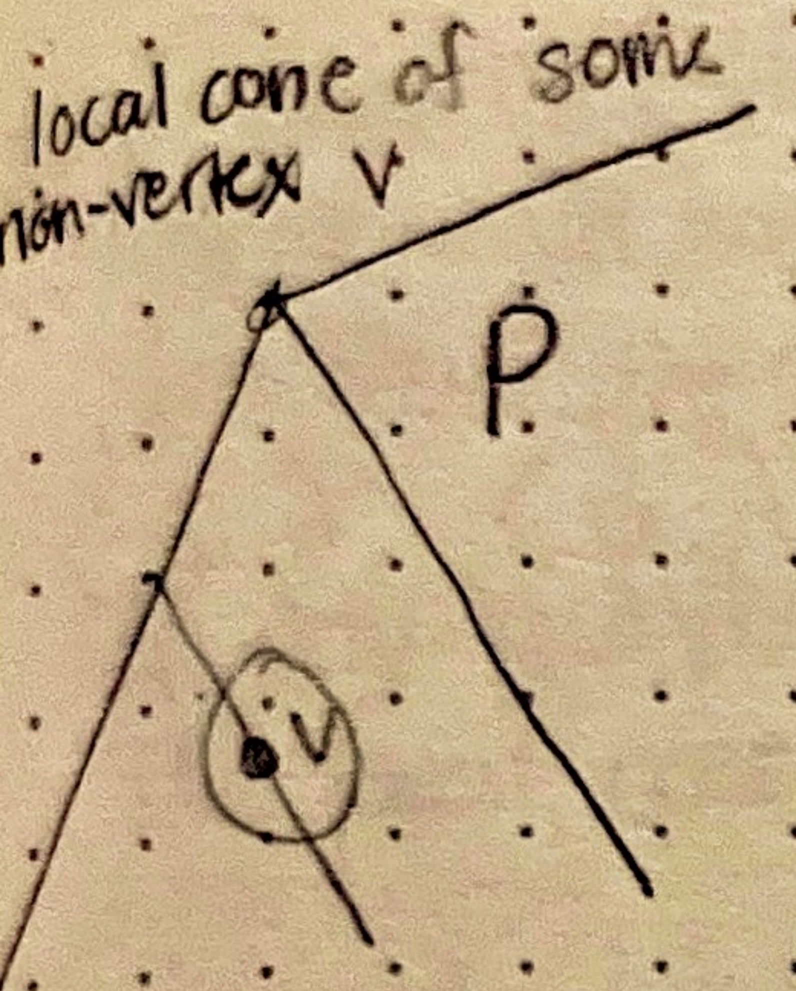




$\underline{\text{Proof}}$ of thrm 2: $[P] = \sum \alpha_i [C_i]$ ← simple cones

$= \sum_v \Big(\underbrace{\sum_j \beta_{v_j} [C_{v_j}]}_{\equiv \bmod A'} \Big)$ ← all cones have the same vertex v.

$C_{v,P}$ the local cone of P at v



$\equiv \sum_v [C_{v,P}]$

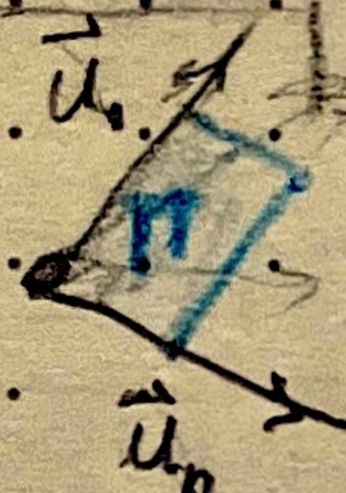
$\overset{\bmod A'}{\equiv} \sum_{v \text{ vertex of } P} [C_{v,P}]$

$\underline{\text{Proof}}$ of Thrm 1:

$\underline{\text{uniqueness}}$ of map S.

Last lecture, saw that (1), (2) $\Rightarrow$ $\boxed{S(C) = \dfrac{\sum_{b \in \Pi \cap \mathbb{Z}^n} x^b}{\prod_{i=1}^{n} (1 - x^{u_i})} \quad (**)}$

simple cone generated by vectors $\vec{u}_1, \ldots, \vec{u}_n$



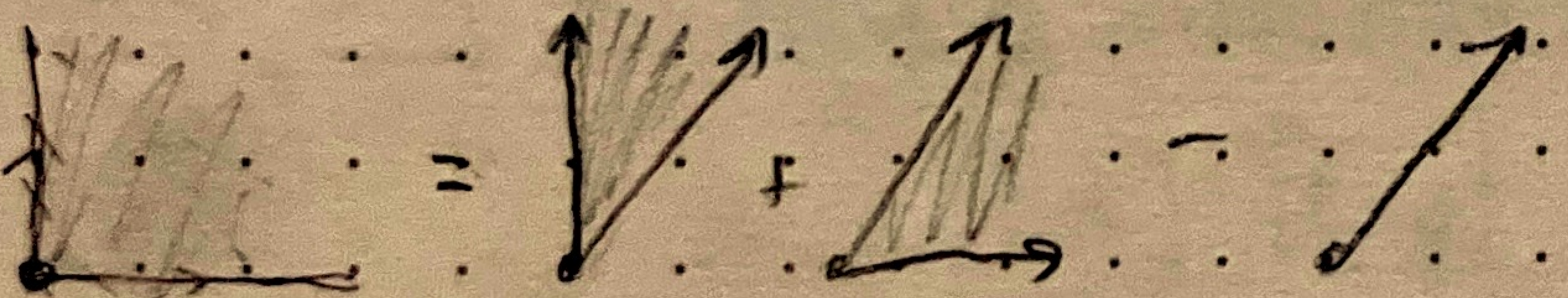
Together w/ fact that anything can be generated by simple cones, this gives us uniqueness.

$\underline{\text{existance}}$ of S

NTS. if $[P] = \sum \alpha_i [C_i] = \sum \alpha_i' [C_i']$ ← simple cones (two different ways to write $[P]$ as lin. combo of simple cones)

then the corresp. sums of rational expressions (**) are the same.

$\underline{\text{E.x.}}$



$\dfrac{1}{(1-x)(1-x)} = \dfrac{1}{(1-y)(1-xy)} + \dfrac{1}{(1-x)(1-xy)} - \dfrac{1}{(1-xy)}$

→ Left as pset exercise to prove.

Fix a $\underline{\text{generic}}$ vector $\vec{h} = (h_1, \ldots, h_n) \in \mathbb{R}^n$

↳ $\langle \vec{h}, \vec{u}_{iv} \rangle \neq 0 \;\; \forall\, i, v$.

$\vec{x} = e^{t\vec{h}} = (e^{th_1}, \ldots, e^{th_n}) \quad t \in \mathbb{R}$.

Now we want to find $\#(P \cap \mathbb{Z}^n)$ by plugging in $\vec{x} = (1, 1, \ldots, 1)$ into (*), but we can't do this directly, so insted we define a convenient function for $\vec{x}$ and take the limit as it goes to $(1, 1, \ldots, 1)$

Plug in $\vec{x} = e^{t\vec{h}}$ into (*). Note as $t \to 0$ $x \to (1, \dots, 1)$.

Then $\#(P \cap \mathbb{Z}^n) = \lim_{t\to 0} \left(t^n \sum_{v \text{ vertex}} \left(\sum_{b \in \Pi_v \cap \mathbb{Z}^n} e^{t\langle \vec{h}, \vec{b} \rangle} \right) \prod_{i=1}^{n} \frac{1}{1 - e^{t\langle \vec{h}, \vec{u}_{iv} \rangle}} \right)$

calculate using L'hôpital's rule

$$\rightsquigarrow \frac{t}{1-e^{ct}} = -\sum_{n=0}^{\infty} B_n c^{n-1} \frac{t^n}{n!} = -c^{-1} + \frac{1}{2} c^{-2} t - \dots$$

Bernoulli numbers

n	0	1	2	3	4	5	6	7	...
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$\frac{-1}{30}$	0	$\frac{1}{42}$	0	...

Also have interesting comb. interpretation (when multiplied by some factor to get integers)

$\rightarrow \lim_{t\to 0} \frac{1}{t^n} \sum_{k \geq 0} a_k t^k$ where a_k depends only on $\langle \vec{h}, \vec{b} \rangle$, $\langle \vec{h}, \vec{u}_{iv} \rangle$ and B_i

Since the limit exists $\Rightarrow a_0 = a_1 = \dots = a_{n-1} = 0$
and a_n = the answer.

Cor of Brion's Thrm:
$$\#(P \cap \mathbb{Z}^n) = \sum_{v \text{ vertex}} [t^n] \left(\left(\sum_{b \in \Pi_v \cap \mathbb{Z}^n} e^{t\langle h, b \rangle} \right) \prod_{i=1}^{n} \frac{t}{(1 - e^{t\langle h, u_{iv} \rangle})} \right)$$

comes from here

$$\mathrm{Vol}\, P = \lim_{k\to\infty} \frac{\#(kP \cap \mathbb{Z}^n)}{k^n}$$

$$= \frac{1}{n!} \sum_{\substack{v \text{ vertex} \\ \text{of } P}} \underbrace{\mathrm{Vol}(\Pi_v)}_{= \#(\Pi_v \cap \mathbb{Z}^n)} \cdot (-1)^n \langle h, v \rangle^n \prod_{i=1}^{n} \frac{1}{\langle h, u_{iv} \rangle}$$

This is now true for any arbitrary polytope.

Next lecture: Will do some corollaries & examples.