LECTURE 27 FRI 11/8

Last lecture "proved"

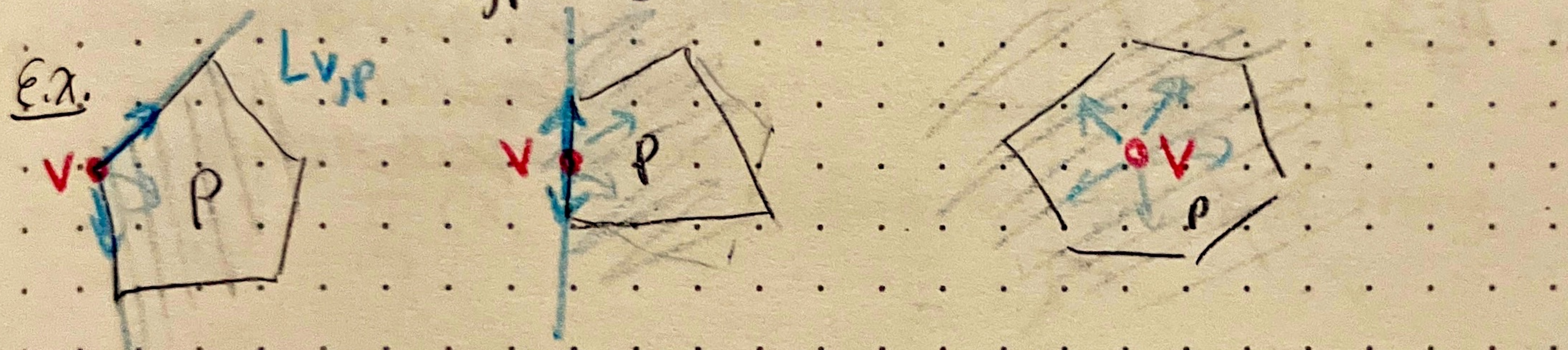
<u>Thm 2</u>: $[P] \equiv \sum_{v \text{ vertex of } P} [L_{v,P}]$ (mod ruled polyhedra)

Today we'll make this proof clearer

Let $A_{\mathbb{R}}$ be the space of functions $\mathbb{R}^n \to \mathbb{R}$ spanned by characteristic function $[P]$ of arbitrary polyhedra $[P]$ in $\mathbb{R}^n$
(does not need to be rational)

For any polyhedron P and any pt. $v \in P$
The local cone $L_{v,P} = \{ v + t\vec{u} \mid \forall \vec{u} \text{ s.t. } v + \vec{u} \in P \text{ and } t \geq 0 \}$

<u>Ex.</u>



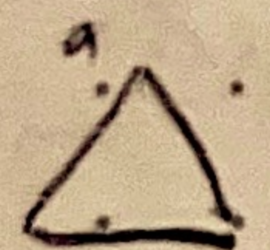
pointed cone half space the whole space

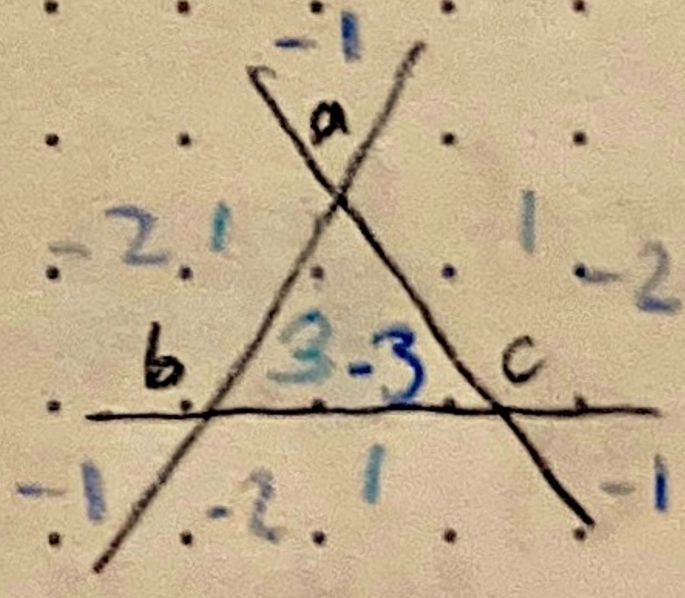
If v on a face F, we can label corresp. half space as $L_{F,P}$.

Q: How to express $[P]$ as a lin. comb. of $[L_{F,P}]$

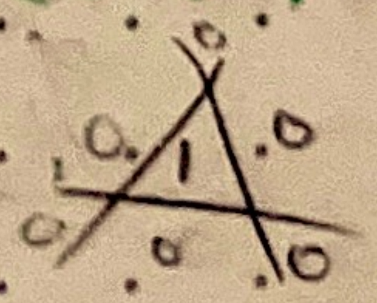
<u>Ex.</u> $P =$ a•—•b, $L_a =$ a•⟶, $L_b =$ ⟵•b, $L_{ab} = \mathbb{R}$

$[P] = [L_a] + [L_b] - [L_{ab}]$

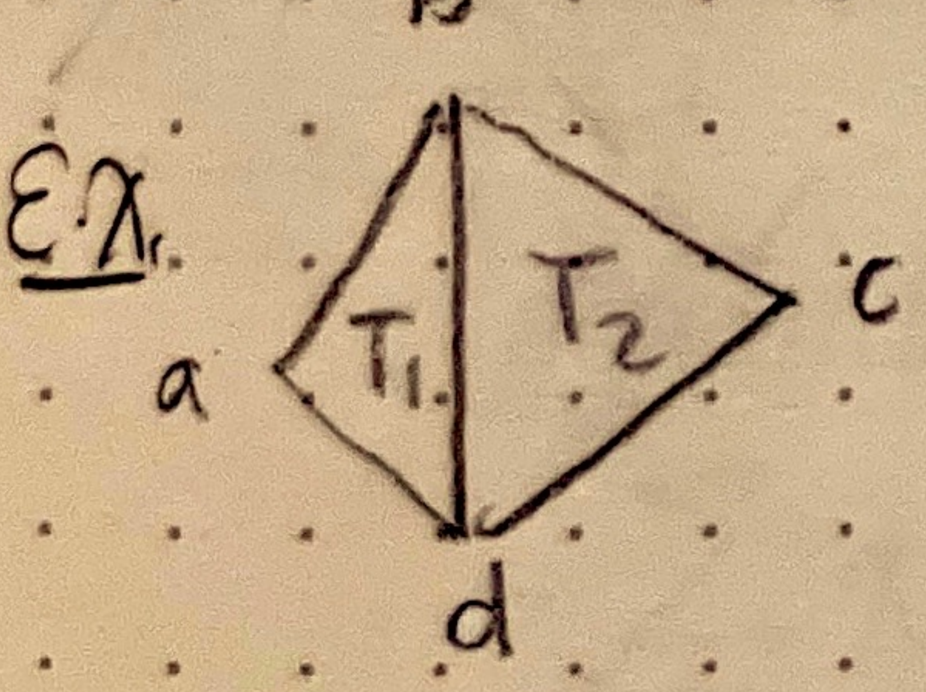
<u>Ex.</u> $P =$  $[P] = [L_a] + [L_b] + [L_c]$
$- \underline{[L_{ab}] - [L_{bc}] - [L_{ac}]}$
$+ [L_{abc}]$



+1 to everywhere



<u>Ex.</u>



$P = [T_1] + [T_2] - [\overline{b,d}]$
$= [L_a] + [L_c] + [L_b] + [L_d]$
$- [L_{ab}] - [L_{bc}] - [L_{cd}] - [L_{ad}]$
$- [\cancel{\mathbb{R}^2}] + 2[\mathbb{R}^2]$

Guess: $\forall$ polyhedra P

$$[P] \overset{?}{=} \sum_{F \text{ face of } P} (-1)^{\dim F} [L_{F,P}]$$

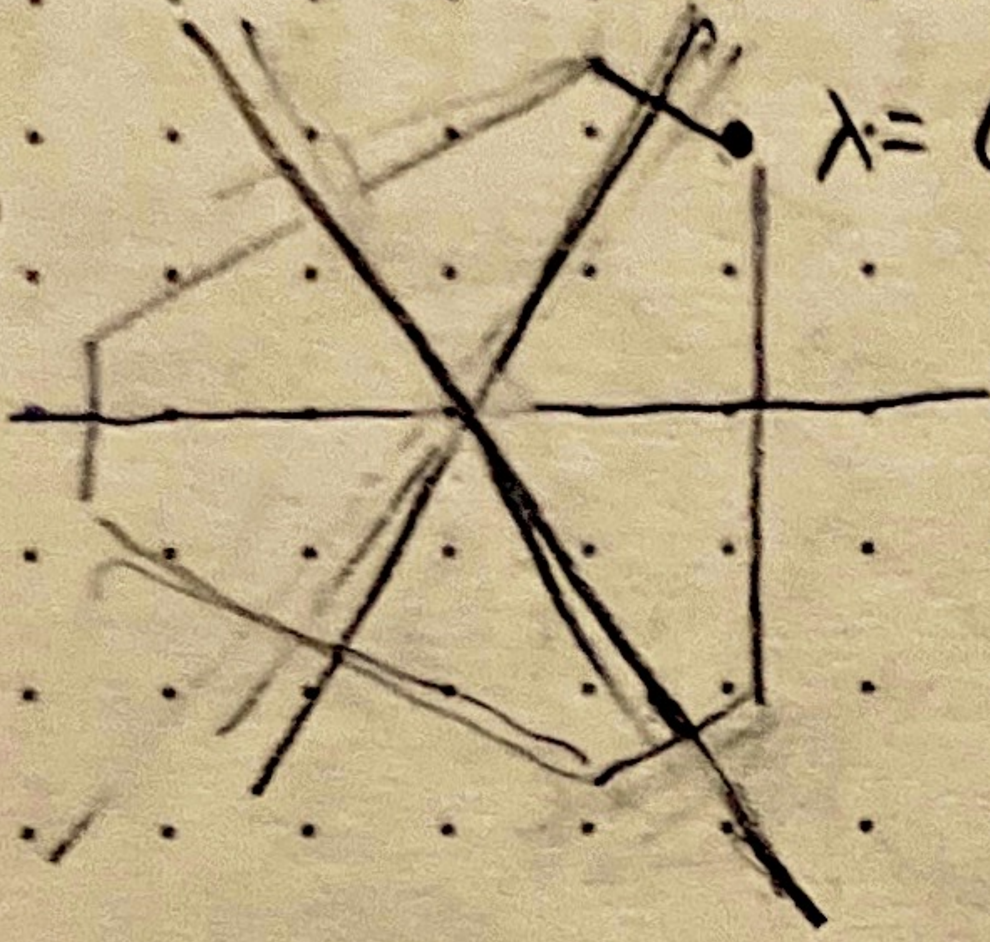
H.W. exercise: Check this and prove or correct it.

Permutohedra

$\lambda = (\lambda_1 \geq \dots \geq \lambda_n) \in \mathbb{Z}^n$ (we can assume that λ is a partition.)

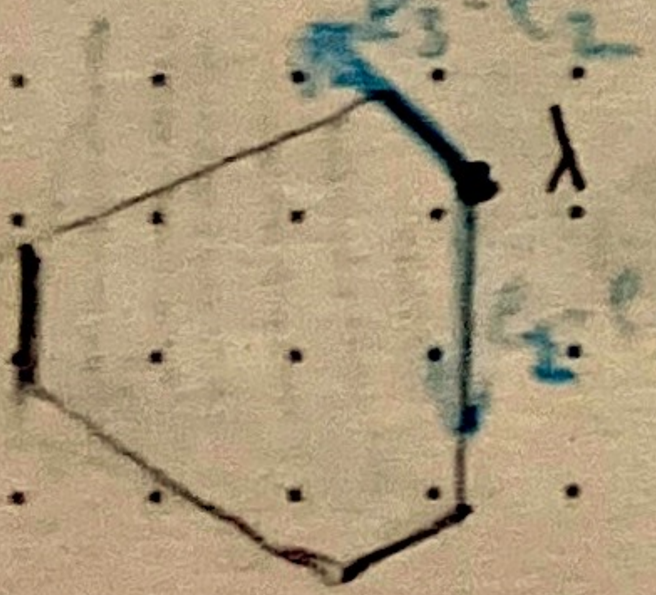
$P(\lambda) := \mathrm{conv}(w(\lambda) \mid w \in S_n)$

Ex. $n=3$



A hexagon if $\lambda_1 < \lambda_2 < \lambda_3$. More degenerate cases are also permutohedra.

If $\lambda_1 > \lambda_2 > \dots > \lambda_n$, then $P(\lambda)$ has $n!$ vertices & is a simple polytope. Combinatorially equiv. to standard permutohedron.



At vertex λ, the generators of $L_{\lambda, P(\lambda)}$ are

$$\vec{e}_{i+1} - \vec{e}_i = (0, \dots, 0, \underset{i}{-1}, \underset{i+1}{1}, 0, \dots, 0) =: u_i \quad i = 1, \dots, n-1$$

Cor. of Brion's Formula:

(1) $$\sum_{\beta \in P(\lambda) \cap \mathbb{Z}^n} x^\beta = \sum_{w \in S_n} w \left(\frac{x^\lambda}{\prod_{i=1}^{n-1} \left(1 - \frac{x_{i+1}}{x_i}\right)} \right)$$

(2) $$\mathrm{Vol}\, P(\lambda) = \frac{1}{(n-1)!} \sum_{w \in S_n} \frac{(\lambda_1 h_{w(1)} + \dots + \lambda_n h_{w(n)})^{n-1}}{\prod_{i=1}^{n-1} (h_{w(i)} - h_{w(i+1)})}$$

In (1) can be any $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n) \in \mathbb{Z}$

In (2) can be any distinct $h_1, \dots, h_n \in \mathbb{R}$

Let's compare (1) with def of Schur polynomials

Def: $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \in \mathbb{Z}^n$ (a partition)

$$S_\lambda(x_1, \dots, x_n) = \sum_{w \in S_n} w \left(\frac{x^\lambda}{\prod_{1 \leq i < j \leq n} (1 - \frac{x_j}{x_i})} \right)$$

Note: Usually s_λ defined classically as

$$S_\lambda = \frac{a_{\lambda+\rho}}{a_\rho} \qquad \rho = (n-1, n-2, \dots, 0)$$

vandermonde-like determinants $\quad a_\lambda = \det \begin{vmatrix} x_1^{\lambda_1} & x_2^{\lambda_1} & \cdots & x_n^{\lambda_1} \\ \vdots & & & \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \cdots & x_n^{\lambda_n} \end{vmatrix}$

$$\frac{x^\lambda}{\prod_{i<j} (1 - \frac{x_j}{x_i})} = \frac{x^{\lambda+\rho}}{\prod_{i<j} (x_i - x_j)}$$ so formulas give same thing

G-Schur polynomials

$G = (V, E)$ a graph on vertex set $V = [n]$.

Def: $S_\lambda^G(x_1, \dots, x_n) := \sum_{w \in S_n} w \left(\frac{x^\lambda}{\prod_{\substack{(i,j) \in E \\ i<j}} (1 - \frac{x_j}{x_i})} \right)$ $\quad \lambda = (\lambda_1, \dots, \lambda_n)$ a partition.

Ex. 1.) $G = K^n$ get usual Schur polys

$$S_\lambda = \sum_{\beta \in \mathbb{Z}^n} K_{\lambda\beta} x^\beta$$

Kostka numbers $= \#$ SSYT of shape λ & content β.

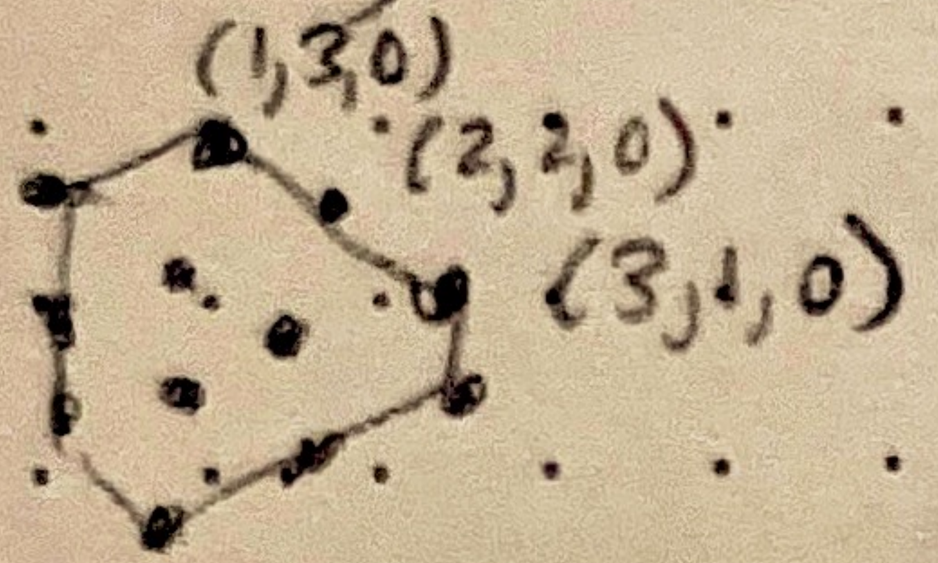
$K_{\lambda\beta} \neq 0$ iff $\beta \in P(\lambda) \cap \mathbb{Z}^n$.

2.) $G = \overset{1}{\bullet} - \overset{2}{\bullet} - \overset{3}{\bullet} - \cdots - \overset{n}{\bullet}$

$$S_\lambda^G = \sum_{\beta \in P(\lambda) \cap \mathbb{Z}^n} x^\beta$$ flattened Schur poly

$$= \sum_{\substack{\beta \text{ s.t.} \\ K_{\lambda\beta} \neq 0}} x^\beta$$ call these S_λ^{Fl}

Ex. $n=3$, $\lambda=(3,1,0)$



$S_{(3,1,0)}$

$= x_1^3 x_2 x_3^0$ + all perms. $\begin{array}{|c|c|c|}\hline 1&1&1\\ \hline 2& \\ \cline{1-1}\end{array}$

$+ x_1^2 x_2^2 x_3^0$ + all perms $\begin{array}{|c|c|c|}\hline 1&1&2\\ \hline 2& \\ \cline{1-1}\end{array}$

$+ 2x_1^2 x_2 x_3$ $\begin{array}{|c|c|c|}\hline 1&1&2\\ \hline 3& \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2& \\ \cline{1-1}\end{array}$

$= S^{f1}_{(3,1,0)} + S^{f1}_{(2,1,1)}$

3.) $G = \overset{\bullet}{1}\ \overset{\bullet}{2}\ \cdots\ \overset{\bullet}{n}$ (the empty graph)

S^G_λ = rescaled monomial symmetric polynomial m_λ.

<u>Thm</u>: For any fixed graph G, $\{S^G_\lambda\}$ is a lin. basis of $\Lambda_n = \mathbb{R}[x_1, \ldots, x_n]^{S_n}$.

<u>Problem</u>: How to relate two bases $\{S^{G_1}_\lambda\}$ and $\{S^{G_2}_\lambda\}$? For which pairs of graphs (G_1, G_2) do we get non-negative basis exchange basis matrix?