LECTURE 29 Fri 11/15

Next Week

Monday: Some of Prof. Postnikov's students will be speaking
Wednesday } 2 guest lectures
Friday } by Colin Defant

Hypersimplices

$$\Delta_{k,n} := \mathrm{conv}(w(\underbrace{1,\dots,1}_{k}, \underbrace{0,\dots,0}_{n-k}) \mid w \in S_n\}$$

$$= [0,1]^n \cap \{\sum_{i=1}^{n} x_i = k\}$$ kth section of n-cube

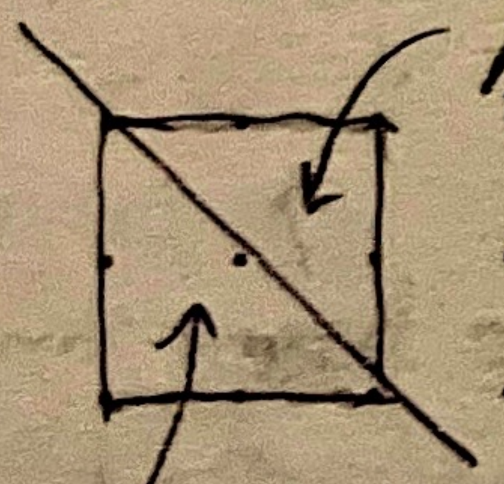
Also $\tilde{\Delta}_{k,n} := [0,1]^n \cap \{k-1 \leq \sum_{i=1}^{n} x_i \leq k\}$ kth slice of n-cube

$\tilde{\Delta}_{k,n}$ = the projection of $\Delta_{k,n+1}$ under

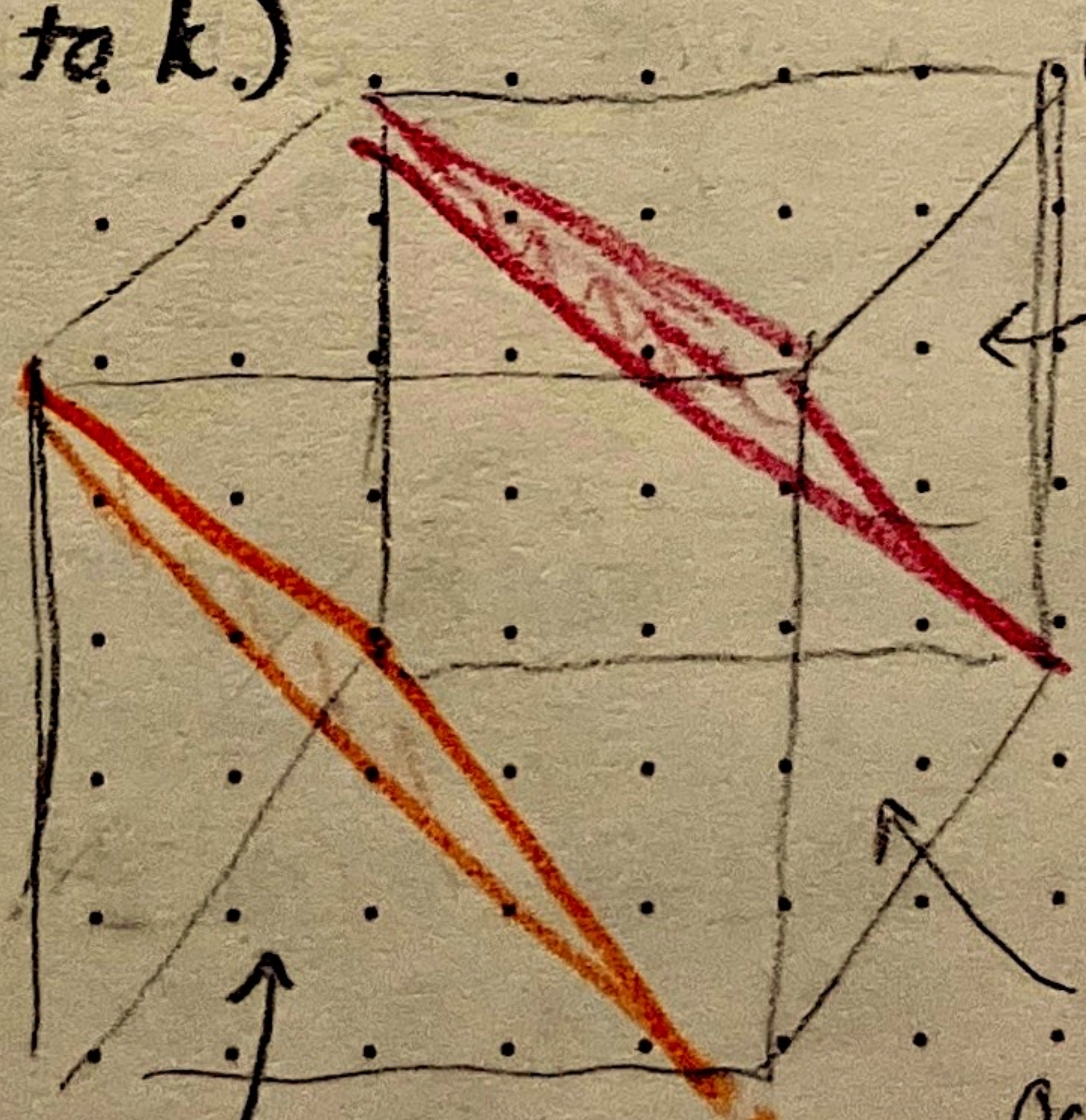
$$p: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$$
$$(x_1, \dots, x_{n+1}) \longmapsto (x_1, \dots, x_n)$$

Why: Because first n coordinates add to some number between $k-1$ & k (and last one takes us up to k)

<u>Ex.</u> $n=2$



<u>Ex.</u>



<u>Thrm</u>: $n!\,\tilde{\Delta}_{k,n} = A_{k-1,n}$ Eularian numbers

$$:= \#\{w \in S_n \mid \mathrm{des}(w) = k-1\}$$

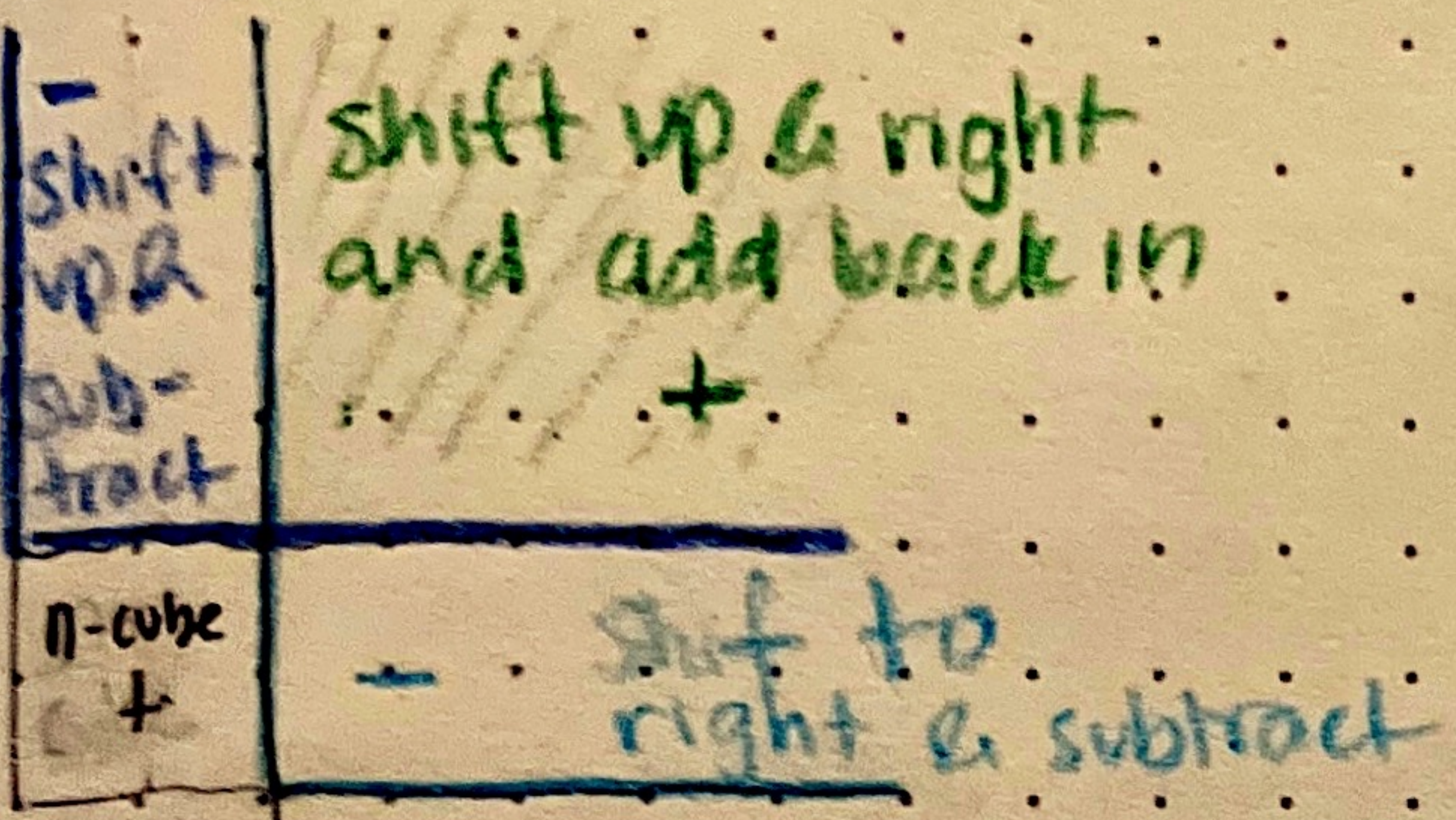
Where $\mathrm{des}(w) := \#\{i \mid w_i > w_{i+1}\}$

~Laplace 1886 (but in terms of probabilities, not polytopes or descents)
Stanley 1977 very nice short combinatorial proof

Ex. $n=3$.

123	des $=0$
132, 231, 213, 312	des $=1$
321	des $=2$

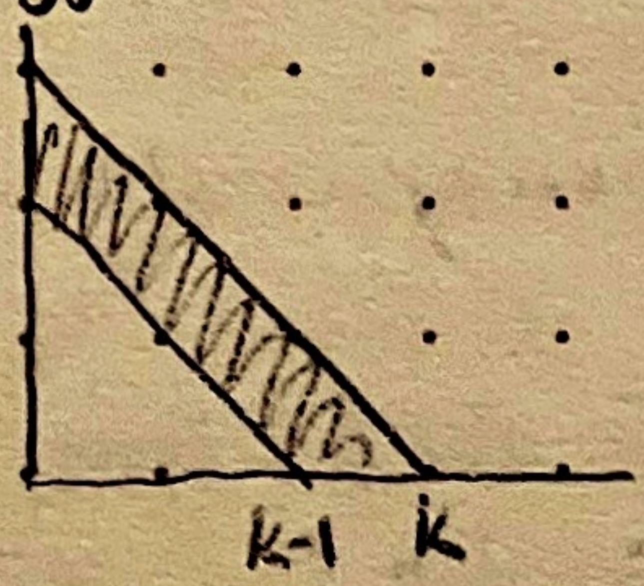
Proof: <u>1st method</u>:
<u>Inclusion-Exclusion</u>



$$[0,1]^n = [C] - \sum_i [C+\vec{e}_i] + \sum_{i,j} [C+\vec{e}_i+\vec{e}_j] - \dots$$

$C_k = k^{th}$ slice of C

$n!\,\mathrm{Vol}\,C_k = k^n - (k-1)^n$ = $n! \times$ Volume of this strip



In each of these summands, it's easy to count the volume of the k^{th} slice because it extends (as above)

$$n!\,\mathrm{Vol}(\tilde{\Delta}_{k,n}) = (k^n-(k-1)^n) - \sum_{i=1}^{n}((k-1)^n-(k-2)^n) + \sum_{i<j}((k-2)^n-(k-3)^n) - \dots$$

shift the strip based on how many e_i's we added.

$$= \sum_{j=0}^{k-1} (-1)^j \binom{n}{j}((k-j)^n-(k-j-1)^n)$$

$\overset{?}{=}$ # perms w/ $k-1$ descents

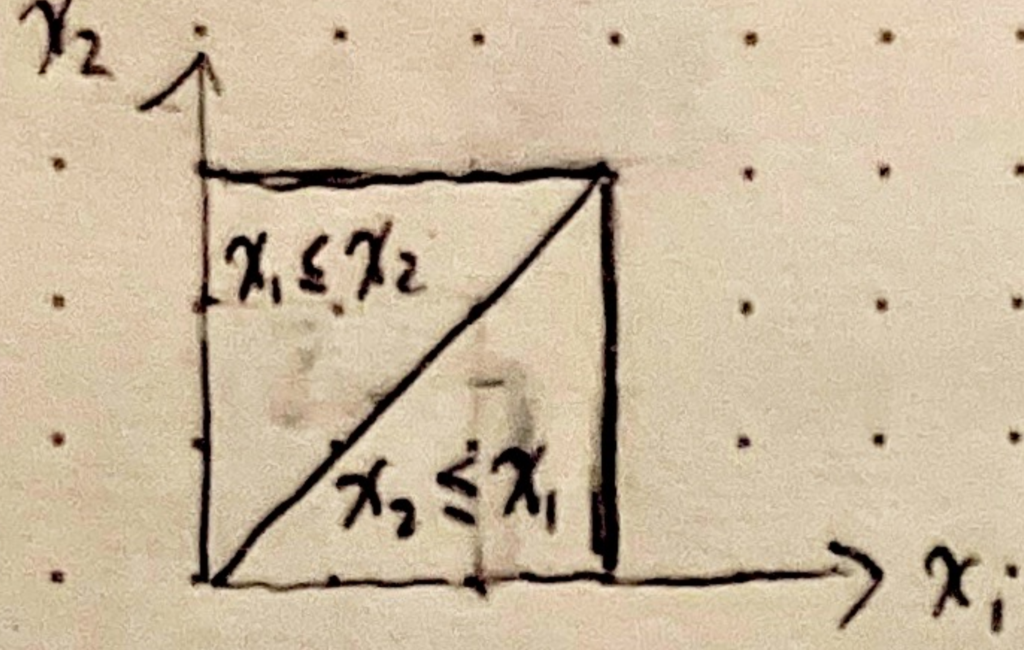
<u>2nd method</u>: Let's do this more combinatorially!

Stanley: How to construct a triangulation of $\tilde{\Delta}_{k,n}$ into $A_{k-1,n}$ simplices.

$\bigcup_k \tilde{\Delta}_{k,n} = [0,1]^n$ $\qquad \sum_k A_{k-1,n} = n!$

So want to triangulate whole cube into $n!$ simplices.

$$[0,1]^n = \bigcup_{w\in S_n} w(\underbrace{\{(x_1,\dots,x_n) \mid 0\le x_1\le x_2 \le \dots \le x_n \le 1\}}_{=:\nabla_w})$$



<u>Problem</u>: This triangulation is not currently compatible with our slicing.

<u>Question</u>: How can we modify it so it is?

Map: $\varphi: [0,1]^n \longrightarrow [0,1]^n$

$(x_1,\dots,x_n) \longmapsto (y_1,\dots,y_n)$

$$y_i = \begin{cases} x_i - x_{i-1} & \text{if } x_i \ge x_{i+1} \\ x_i - x_{i-1}+1 & \text{if } x_i < x_{i+1} \end{cases}$$

For $i=1,\dots,n$. Assume $x_0=0$.

<u>Lemma</u>: φ is a bijective, volume-preserving, piecewise-linear map from the cube to itself which is linear on each simplex ∇_w

<u>Proof</u>:
- Linear on simplices follows directly b/c have same x_i, x_{i+1} relative order $\forall$ pts in ∇_w giving us the same lin. formula
- Volume preserving b/c on each area of linearity has determinant 1
- Bijective follows from inverse map $x_i = (y_1+\dots+y_i) - \lfloor y_1+\dots+y_i \rfloor$

In fact linear on full set of ∇_w that have descents in all the same spots as each other. ∎

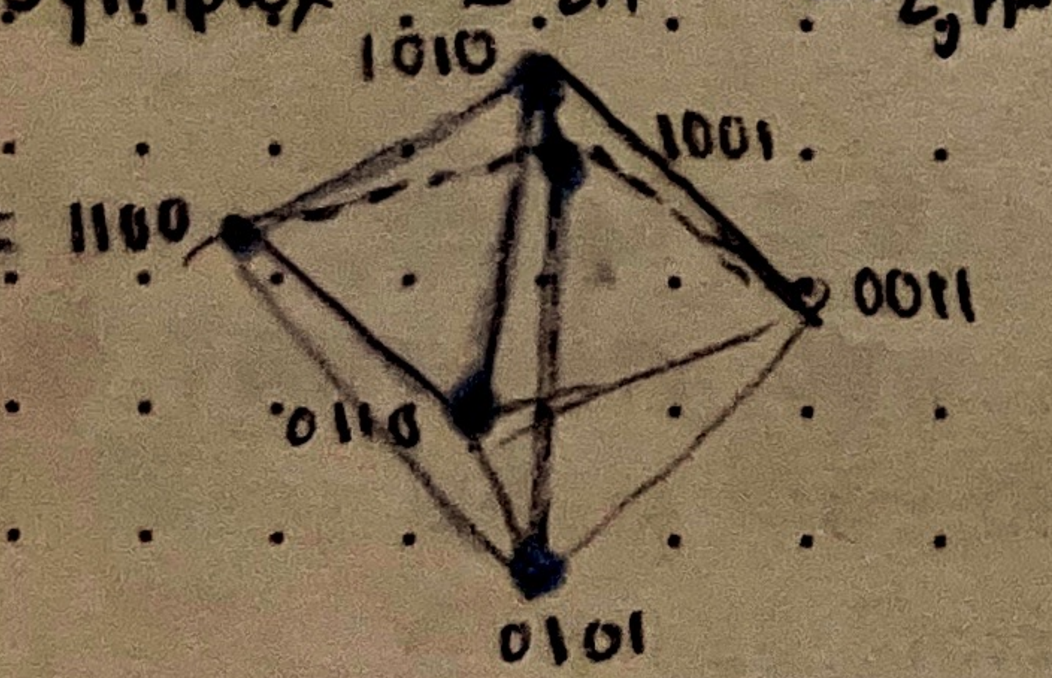
New Triangulation $\varphi(\Delta_w)$ is compatible w/ our slicing:

$$y_1+\dots+y_n = (x_1 - \overset{0}{x_0}) + (x_2 - x_1 (+1)) + (x_3 - x_2 (+1)) + \dots + (x_n - x_{n-1} (+1))$$
$$= \underset{[0,1]}{x_n} + \operatorname{des}(w)$$

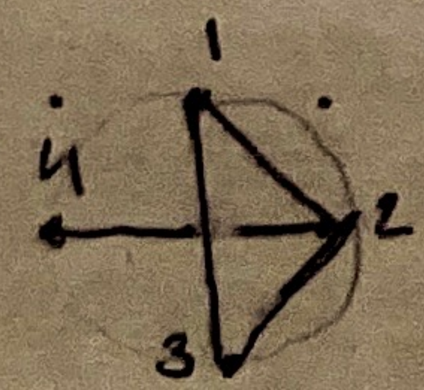
So $\varphi(\nabla_w) \subset k^{th}$ slice where $k = \operatorname{des}(w)+1$

$\Rightarrow$ <u>Thrm</u>: $\bigcup_{\substack{w \text{ with} \\ k-1 \text{ descents}}} \varphi(\Delta_w)$ is a subdivision of $\tilde{\Delta}_{k,n}$ Q.E.D.

2^{nd} hypersymplex $\Delta_{2n} \simeq \tilde{\Delta}_{2,n-1}$

$\Delta_{2,4} =$ 

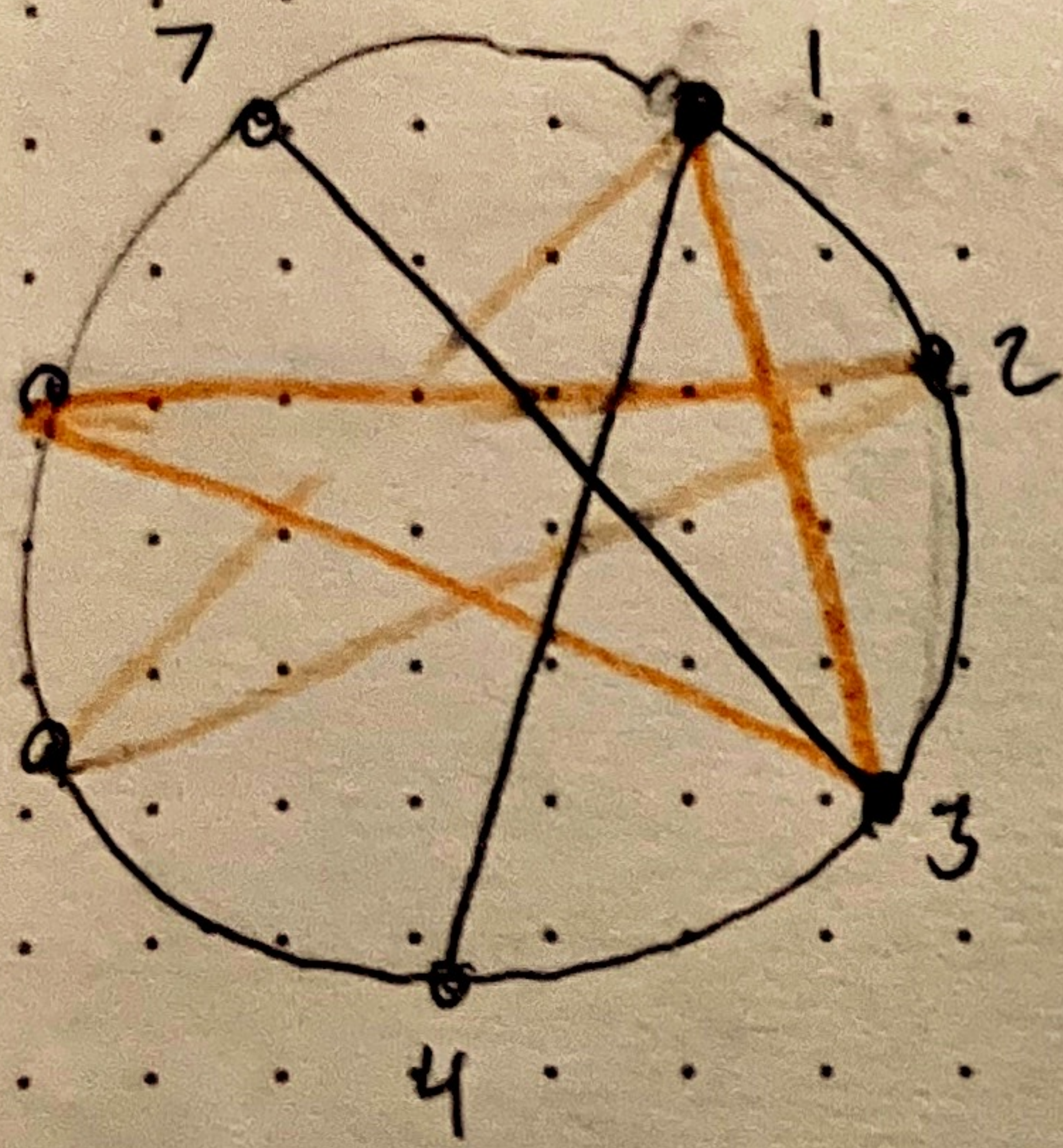
simplicies are conv(1010, 0101, 1100, 0110) and any cyclic rotation of coords

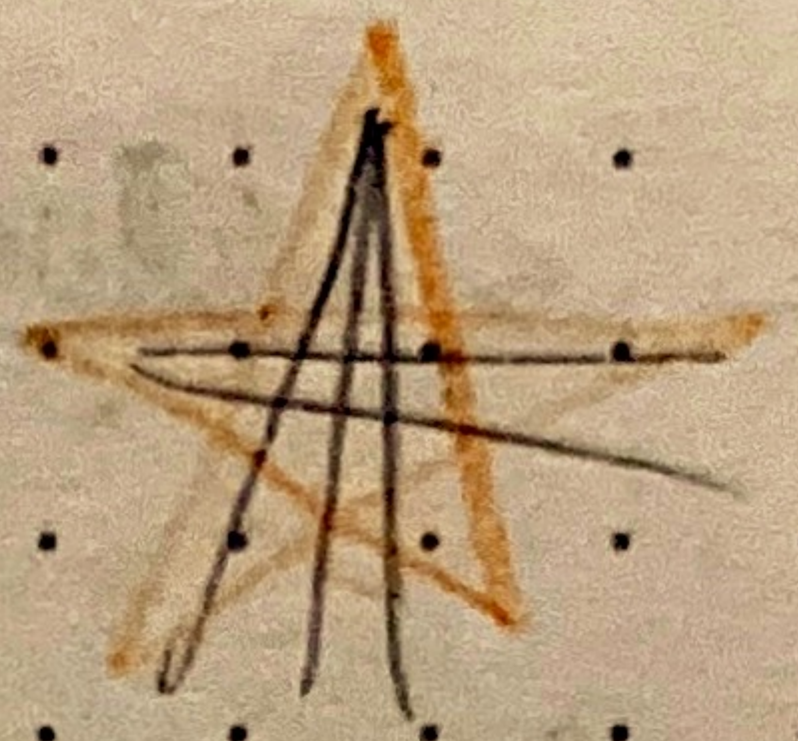
$G =$ 

$\leftarrow$ This is a <u>thrackle</u>

Def: A *thrackle* is a graph G on vertices $1, \dots, n$ arranged on a circle clockwise s.t. any two edges intersect or have a common vertex.

Ex.





Exercise: Any thrackle has at most n edges (where $n = \#$ vertices). If it has n edges, then it is obtained by taking a complete star & adding some leaves

Exercise: Vertices of conv(...) simplices are in bijection w/ maximal thrackles