LECTURE 30 Mon 11/18

Ryota Inagaki:

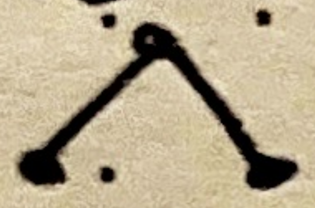
Problem (From Lecture 6 & Pset 1):

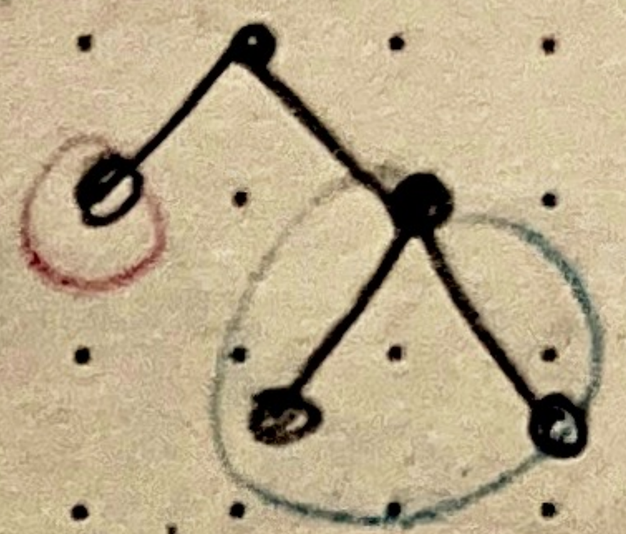
Show #{binary trees w/ n edges & k left edges}
= #{Dyck paths with $2n$ steps & $k+1$ peaks}
= $N(n, k+1)$ Naryana number

Bijection: Recursively defined F.

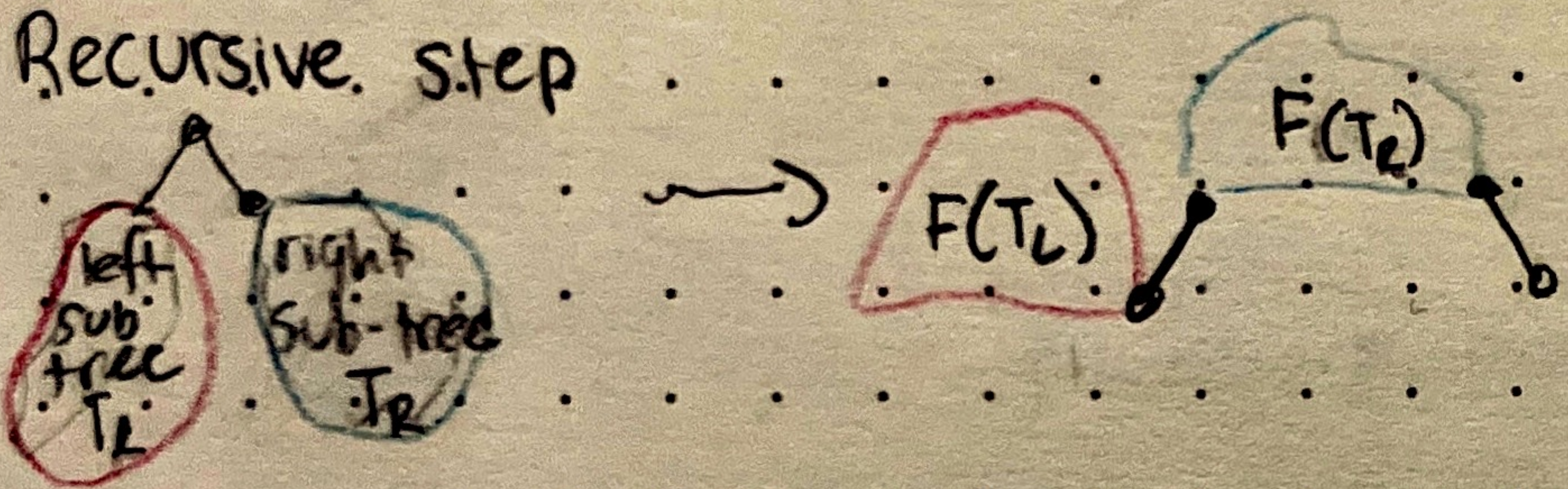
Input tree T

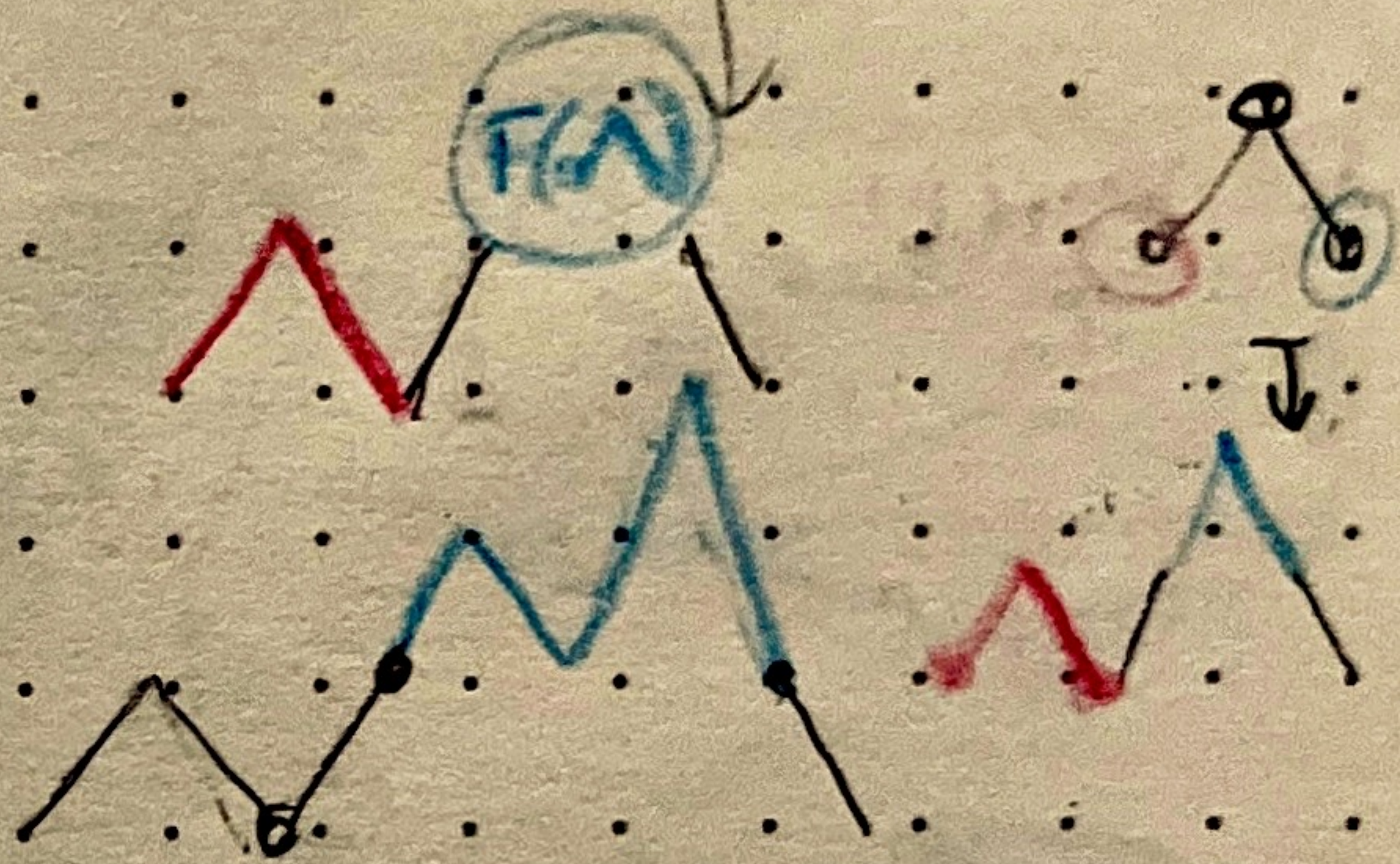
If $T = \bullet$

return 

Ex. T = 

Recursive step

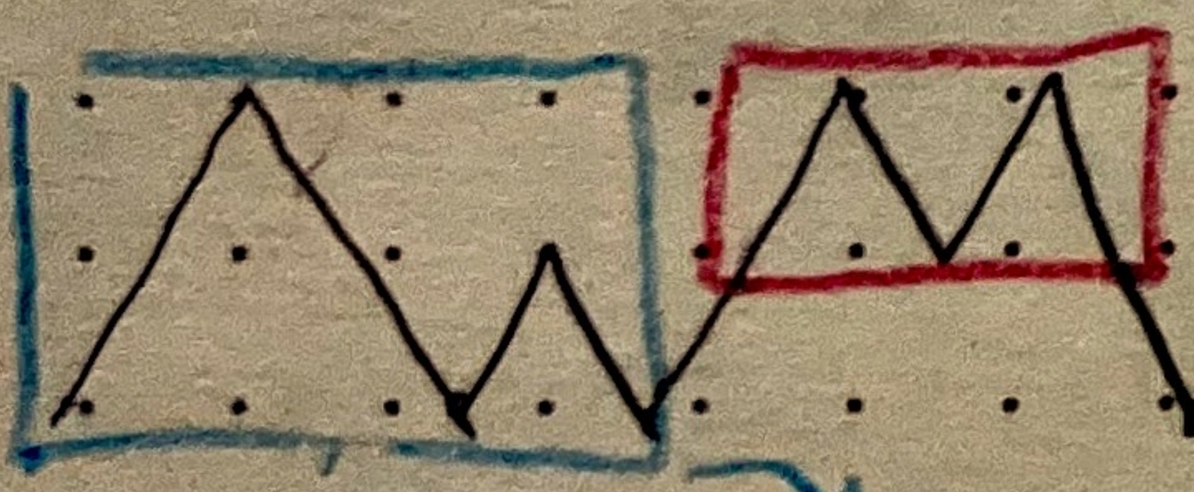




Can show via induction that if input has k left edges, output has $k+1$ peaks

Inverse function F^{-1}

$\Lambda \longrightarrow \bullet$

e.g. 

$F^{-1}(\ \ ) =$ right subtree

F^{-1}(until 2nd to last time we hit 0) = left subtree

Dora Woodruff: Another way to find volume of permutohedron

Mixed Volume:

"mixed volume"

Recall

Prop: There is a unique function $\mathrm{Vol}(Q_1, \dots, Q_n)$ taking in tuples of polytopes $Q_i \subset \mathbb{R}^n$ s.t.

$$\mathrm{vol}\left(\sum_{i=1}^{m} y_i R_i\right) = \sum_{\substack{(i_1, i_2, \dots, i_n) \\ \text{ordered tuples}}} \mathrm{Vol}(R_{i_1}, \dots, R_{i_n})\, y_{i_1} y_{i_2} \cdots y_{i_n}$$

Bernstein's Thm:

n: $\mathrm{vol}(Q_1, \dots, Q_n) = \#$ isolated solutions in $(\mathbb{C} \setminus 0)^n$ of

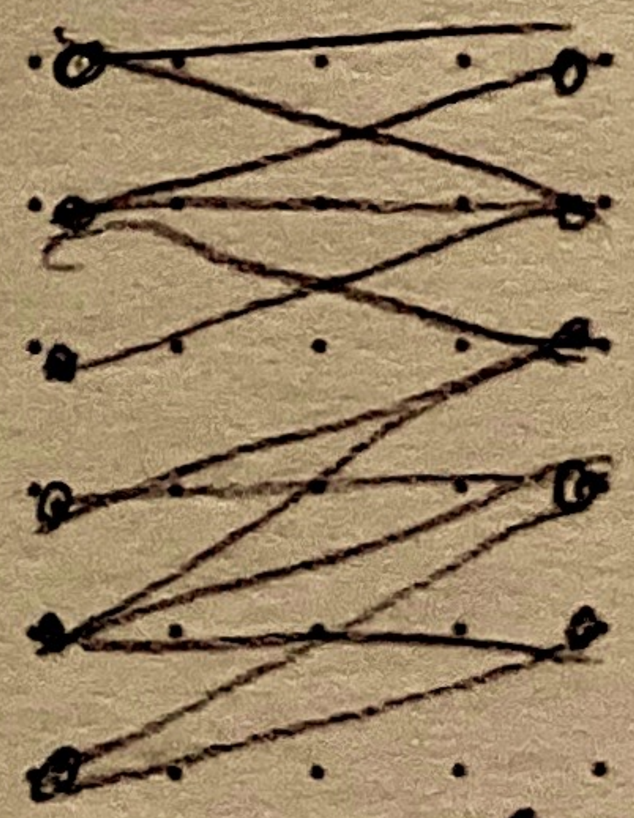
$f_1 = 0$
$\vdots$
$f_n = 0$

where $\mathrm{Newt}(f_i) = Q_i$ (up to scaling)

Thm: $\mathrm{Vol}\left(\sum_{I \subset [n]} y_I \Delta_I\right) = \sum_{(I_{i_1}, \dots, I_{i_{n-1}})} y_{I_1} y_{I_2} \cdots y_{I_{n-1}}$

where $I_1 \dots I_{n-1}$ satisfy Dragon Marriage Condition (DMC) if (equivalently)

1) $|I_1 \cup I_2 \cup \dots \cup I_n| \geq k+1$ Hall's Thm



2) $\forall k$, there is a family of elements $k_j \in I_j$ s.t. $k_j \neq k$, $k_j \neq k_i$

One extra person on the left.
If a dragon takes any one away, can still find a perfect matching

Ex. $\mathrm{Vol}(\Delta_{12}, \Delta_{23})$

$a_1 x_1 + a_2 x_2 = 0$

$a_2 x_2 + a_3 x_3 = 0$

up to scaling assume $x_3 = 1$.

$$\begin{bmatrix} a_1 & a_2 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -b_3 \end{bmatrix}$$

$\Rightarrow$ exactly 1 isolated non-zero sltn.

$\Rightarrow$ Mixed volume = 1.

Use the to calculate $\underbrace{\mathrm{Vol}(P(2,1,0))}_{=\Delta_{12}+\Delta_{13}+\Delta_{23}} = \underbrace{\underbrace{\mathrm{Vol}(\Delta_{12},\Delta_{23})}_{=1} + \dots + \underbrace{\mathrm{Vol}(\Delta_{13},\Delta_{12})}_{=1} + \dots}_{\text{get 6 total}} + \underbrace{\mathrm{Vol}(\Delta_{13},\Delta_{12})}_{=6}$

To solve get coeff matrix $A\vec{x} = \vec{b}$
sltns comes from # $(n-1)\times(n-1)$ minors of A w/ row removed

↳ correspond to sltns of dragon marriage problem

In permutohedron case,

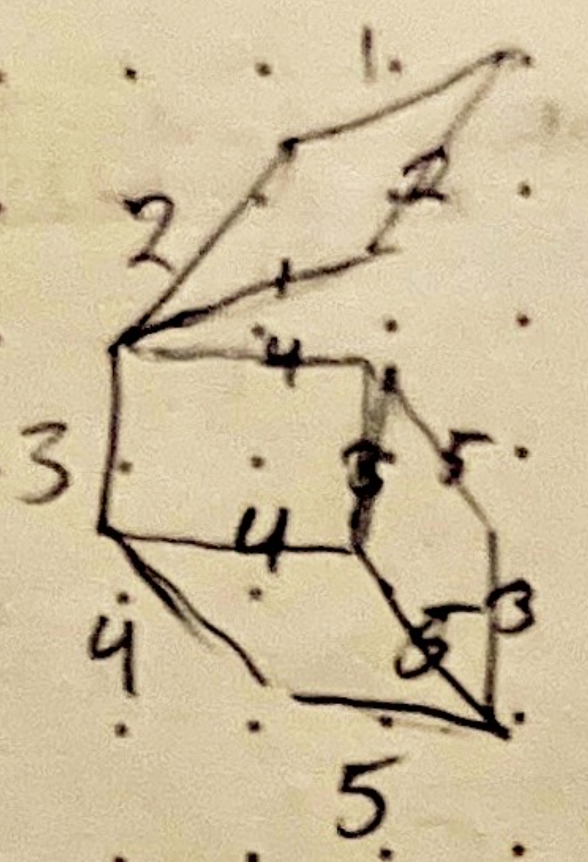
$\mathrm{Vol}\left(\sum_{(i,j)\in E}\right) = $ # spanning trees in G

Ilani Axelrod-Freed: (Hi, it's me your note taker ☺)

Weak Bruhat order and zonotopal tilings

s_i ~ switch the numbers in positions i & $i+1$

Ex. 12345
$\downarrow s_4$
12354
$\downarrow s_1$
21354
$\downarrow s_3$
$21534 \xrightarrow{s_4} 21543 = \sigma$



$Inv(\sigma) = \{(4,5), (3,5), (3,4), (1,2)\}$

Def Inversion set $Inv(w) = \{(w_i, w_j) \mid i<j \text{ and } w_i > w_j\}$

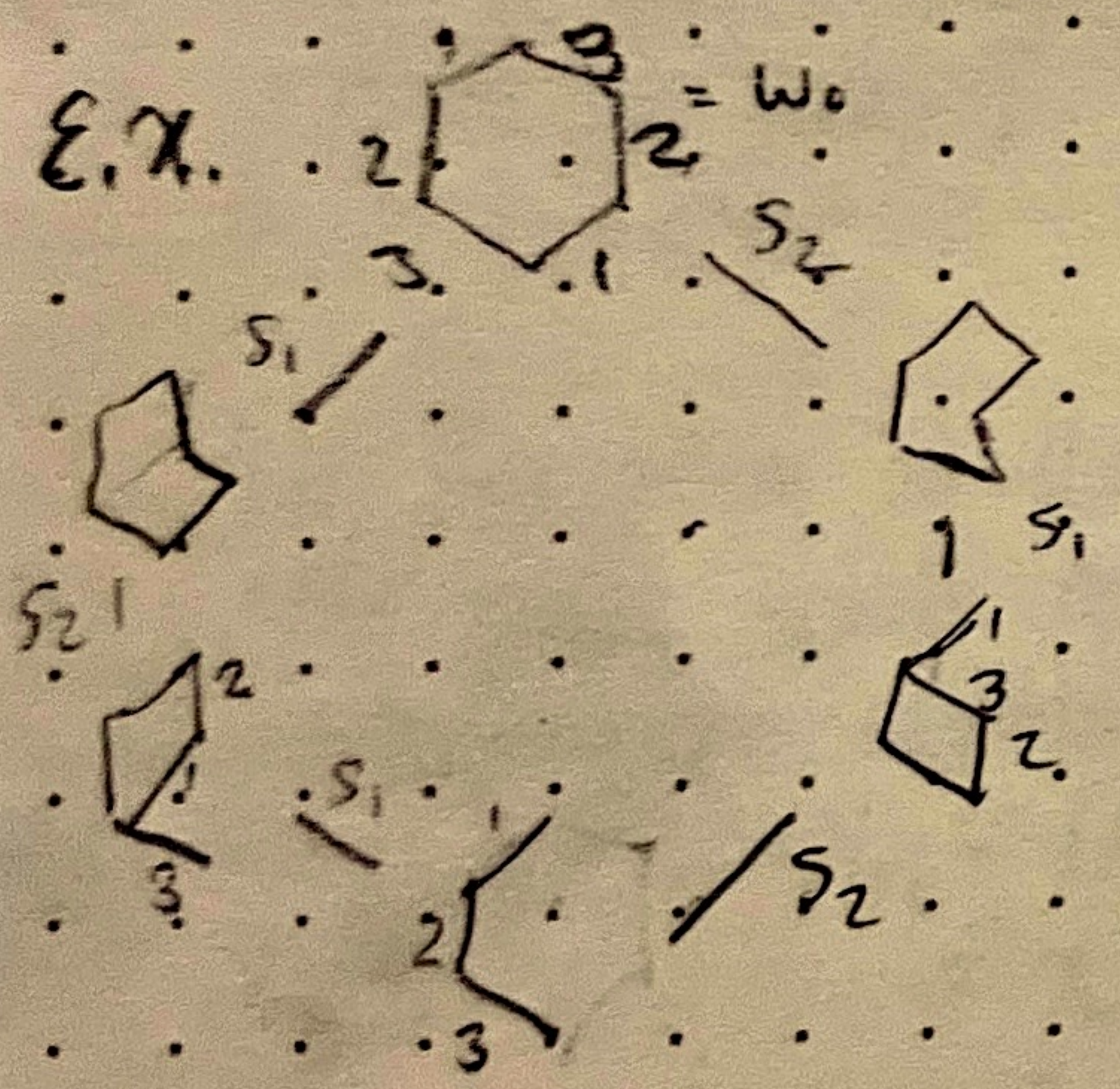
$\Rightarrow$ Exactly the label of our rhombuses!

Def: Weak Bruhat order on S_n

$w <\!\cdot\, w'$ if $w' = s_i w$ and $\text{length}(w') = \text{length}(w)+1$

$\hookrightarrow \#Inv(w')$

Ex.



$w < w' \Rightarrow \text{shape}(w) \subseteq \text{shape}(w')$

zonotopal tilings $\xleftrightarrow{\text{bij}}$ Paths from id to w_0 in Bruhat order mod $s_i s_j = s_j s_i$ for $|i-j| \geq 2$

← can add in either order

Set of inversions satisfies $B(1,n)$

$i<j<k$ jk & ij included → ik included

jk, ik, ij excluded → excluded

Def: Higher Bruhat order $B(k,n)$: $(k-1)$-tuples for "inversions"

For any $i_1 < i_2 < \ldots | \ldots < i_{k+2}$ (in | out)

↑ all but i_1, ↑ all but i_2, ↑ all but i_{k+2}

Can draw line s.t. all $(k+1)$-tuples on one side are included & other are excluded

Set $I <\!\cdot\, I'$ if $I' \supset I$ and has exactly one additional $(k+1)$-tuple.

Elisabeth Bullack:

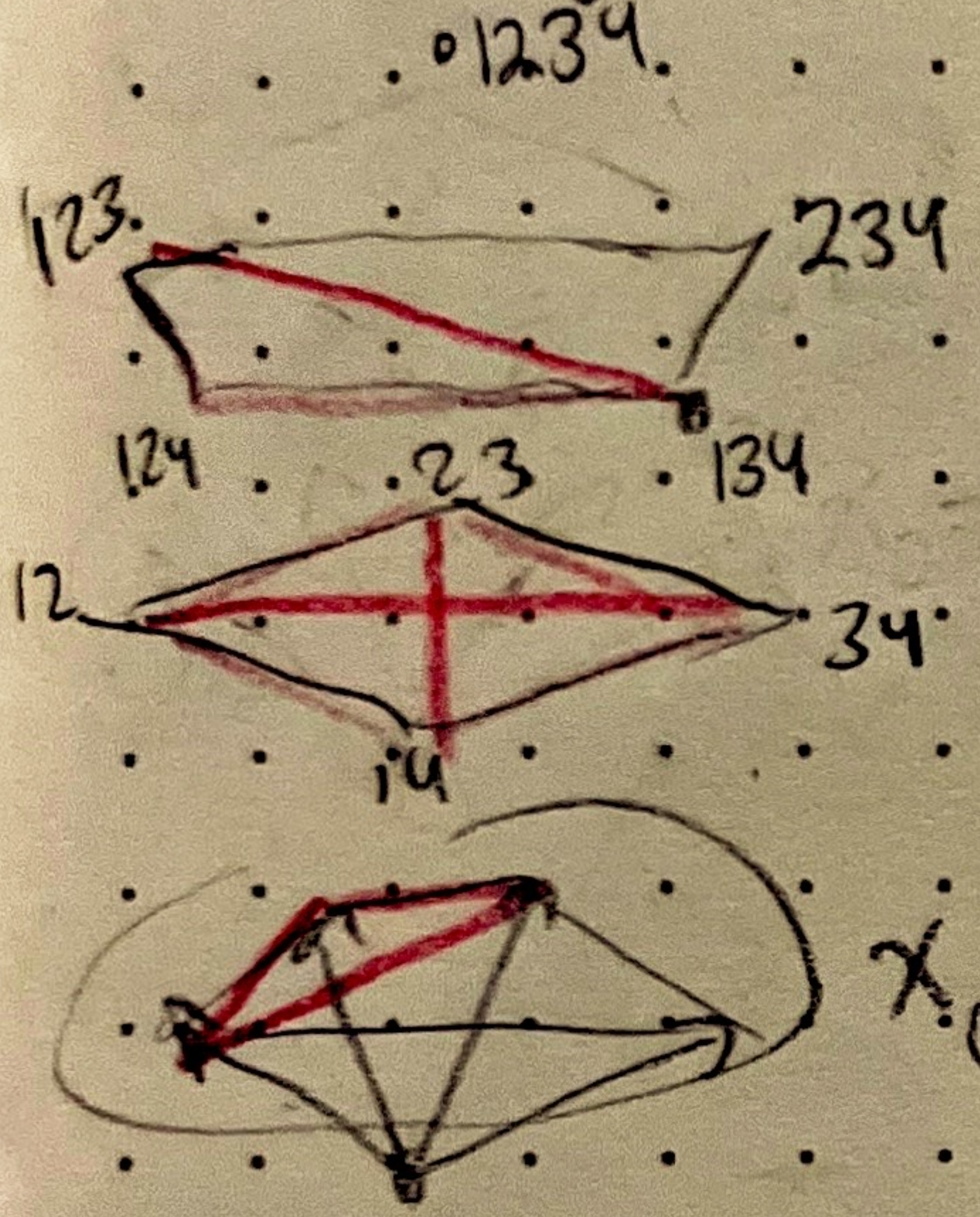
Higher Bruhat orders in terms of zonotopal tilings

Specific zonotope

$Z(n,k) = \sum_{i=1}^{n} [0, v_i]$ $v_i = (1, i, i^2, \dots, i^k)$

Parallelpiped tiles correspond to the $(k+1)$-tuple "inversions"

e.x. $k=3$, $n=4$



cross sections

$x_0 = 1$

2 ways to tile using 4 parallel pipeds.

correspond to adding inversions

all But i_1, all But i_2, ..., all But i_4

in this order

or

this order

Can also define recursively

e.g. each tiling of 2D zonotope corresponds to an elt of $B(2,n)$ and we can give order on them