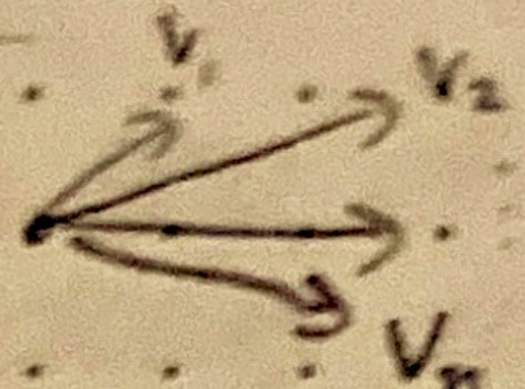
LECTURE 33   Mon 11/25

- Plan: Prof. Postnikov will go over stuff from his talk at the institute (what he was doing last week instead of being here)
- pset 2 will be posted today, due Fri 12/6, presentations Mon 12/9

$$\begin{array}{ccc} \text{Matroids} & \supset & \text{Positroids} \\ \cap & & \cap \\ \text{Polymatroids} & \supset & \text{polypositroids} \end{array}$$

Motivation for matroids

0. Configuration of vectors 

$\in \mathbb{F}^k$

Record the linear (in)dependence data between them

Now will give 3 kryptomorphic def of matroid:

1. Set of bases $M \subseteq \binom{[n]}{k}$ satisfying (Strong) Exchange axiom

usual version

$\forall I, J \in M,\ \forall i \in I$

$\exists\, j \in J$ s.t. $(I \setminus \{i\}) \cup \{j\} \in M$.

strong version

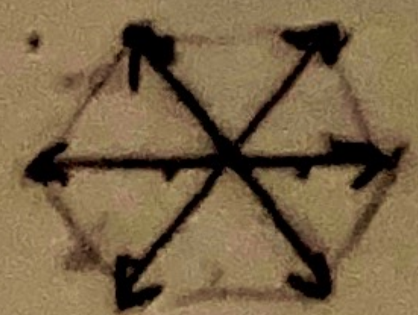
$(J \setminus \{j\}) \cup \{i\} \subset M$ to

Exercise: Prove weak & strong versions equivalent.

2. rank function $\rho : 2^{[n]} \to \mathbb{Z}$
(think of it as counting # lin. ind. vectors in set)
I basis iff $|I| = k = \mathrm{rk}\,M$ and $\rho(I) = k$

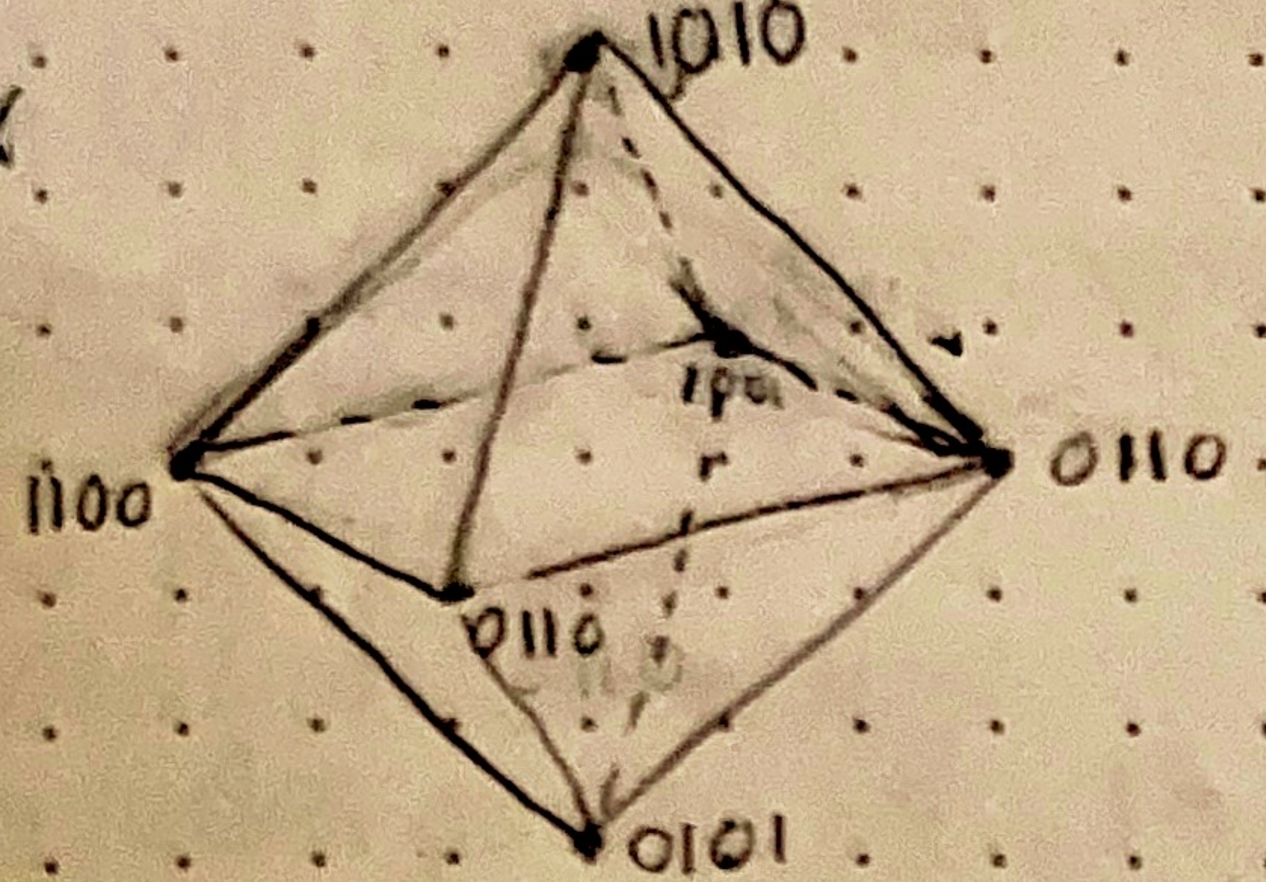
3. Matroid Polytope $P = \mathrm{conv}\left(\sum_{i \in I} e_i \mid I \in M\right)$

Thm GGMS: P is a matroid polytope iff
- all vectors are binary vectors.
- all edges parallel to $e_i - e_j$ (roots) for some i,j

e.x. $n=3$ 

hyersimplex

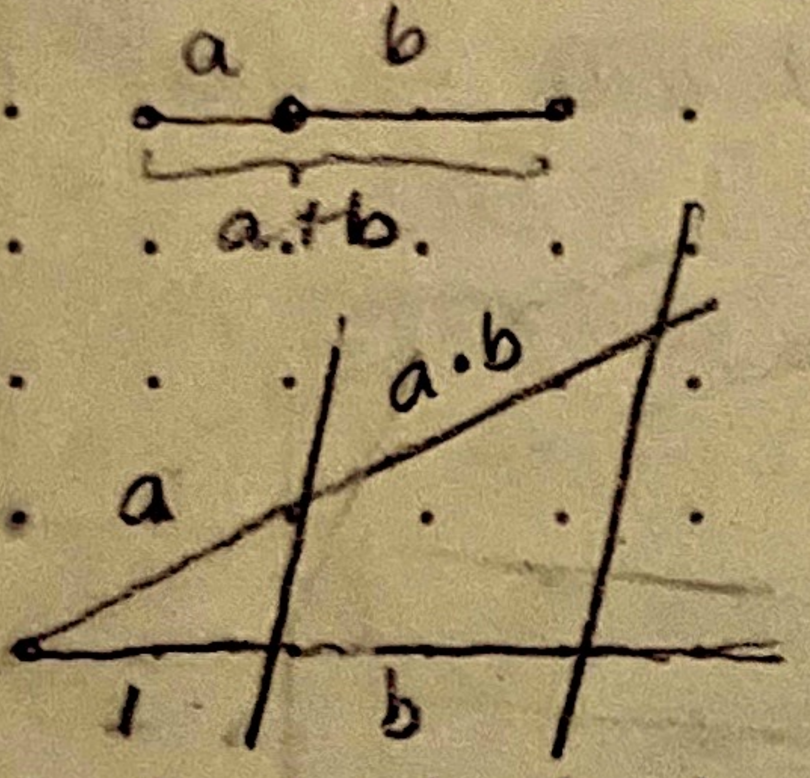
Δ_{24} =



For matroids can take (connected?) parts of this
NOT, for example, the line 1010 —— 0101

Mnev Universality:
Thm: Realization space of M can be arbitrarily complicated
(as bad as anything can be in algebraic geometry)

Eg. Can represent rk 3 matroid as collection of lines on plane
Care if they pass through the same point



addition

0. positroids $:\stackrel{\text{def}}{=}$ matroids realizable by a real matrix

$$A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \text{ s.t. all maximal minors } \Delta_I(A) \geq 0$$

k×n

determinants of k×k submatrices

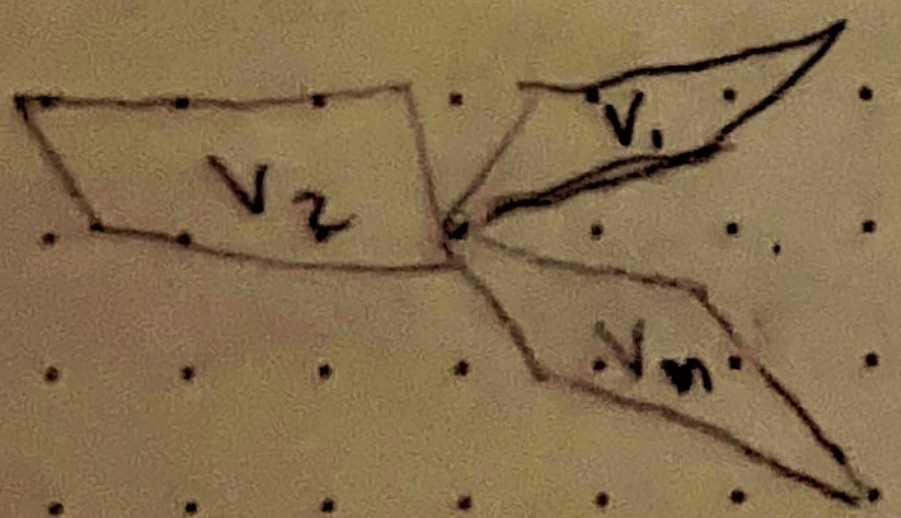
If det > 0 is basis, if det = 0 then not

1. Bases ✓
2. Rank function ✓
3. Matroid polytopes ✓

} (Just add some additional conditions from before)

Polymatroids

0. Motivation: Configuration of linear subspaces (no longer just lines)



Again can give 3 kryptomorphic definitions:

1. <u>M-convex sets</u> $M \subset \mathbb{Z}^n$ (integer vectors, whereas elts of $M \subset \binom{[n]}{k}$ were binary vectors)

$M \subseteq \{x_1 + \dots + x_n = k\} \subset \mathbb{Z}^n$

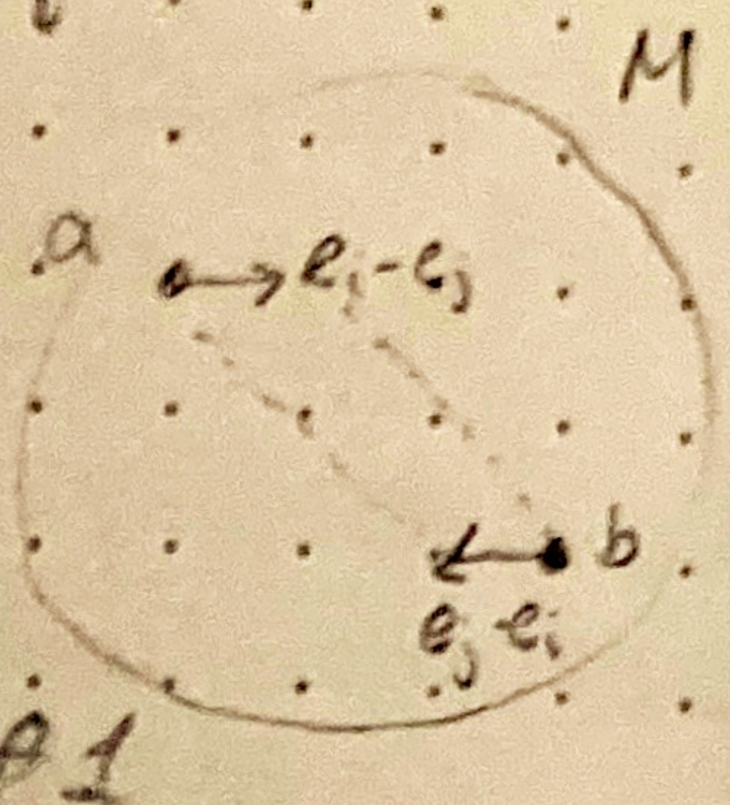
<u>Convexity Condition</u> $\forall\ a, b \in M,\ \forall i$ s.t. $a_i > b_i$

regular exchange axiom

$\exists j$ s.t. $a_j < b_j$ and $a - e_i + e_j \in M$

strong exchange axiom

and $b - e_j + e_i \in M$ too



<u>Note</u>: Specialized to binary vectors this is regular matroid def 1

<u>Exercise</u>: Show regular & strong exchange axioms equiv. here too.

2. <u>Rank function</u> $\rho: 2^{[n]} \longrightarrow \mathbb{Z}$

note: In matroids, when we add 1 elt to subset, rank either stays same or jumps by 1. Here can increase by more

<u>Submodular function</u>: $\forall I, J \subseteq [n]$

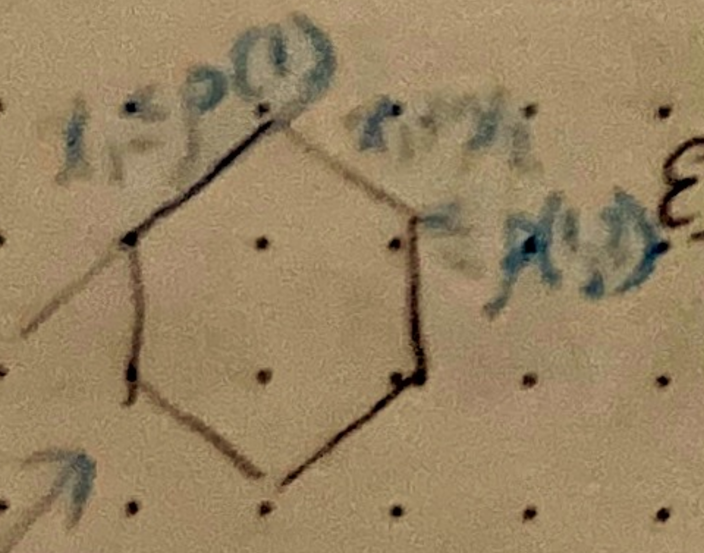
$$\rho(I) + \rho(J) \geq \rho(I \cup J) + \rho(I \cap J)$$

3. generalized permutohedra

polytopes $P \subset \{x_1 + \dots + x_n = c\} \simeq \mathbb{R}^{n-1}$

s.t all edges parallel to $e_i - e_j$. (no longer require vertices to be bin. vectors)

<u>n=3</u>



<u>Ex.</u> std permutohedron

associahedron

graph associahedron

Possible inequalities correspond to sltns to submodular fcns

<u>Polypositroids</u>

0. polymatroids realized by configuration of lin. subspaces s.t. $\exists$ $k \times N$ matrix A s.t. $\Delta_I(A) \geq 0$

$A = \begin{bmatrix} \underbrace{\cdots}_{v_1} & \underbrace{\cdots}_{v_2} & \cdots \end{bmatrix}$ $k \times N$

Practically we can't check this b/c there could be infinitely many basis choices

<u>Problem</u>: Devise a way to check this.