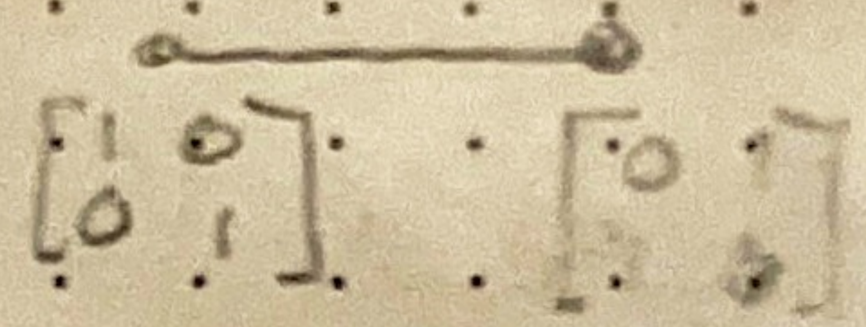
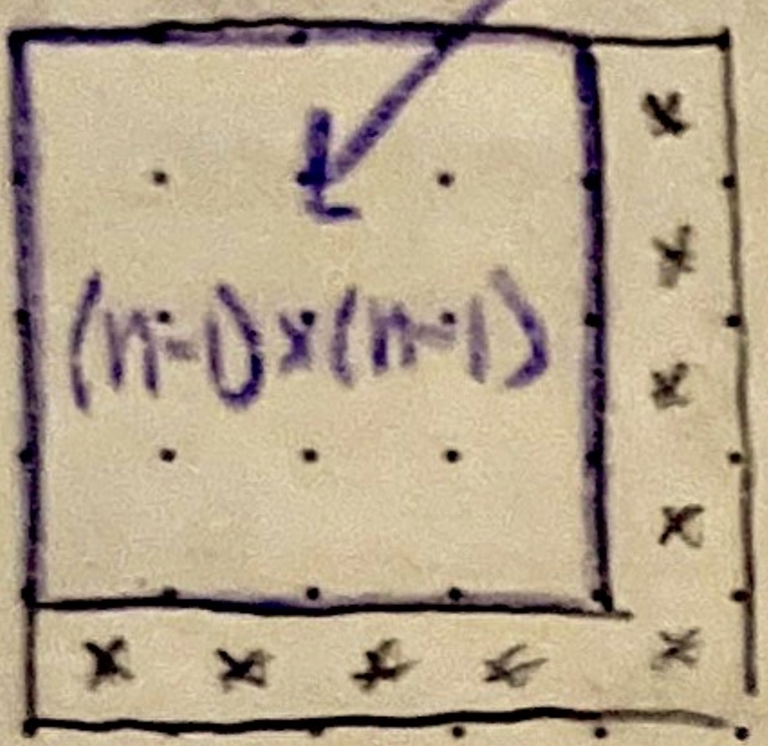
LECTURE 35 Mon 12/2

- Birkhoff polytope B_n
- Matching polytopes
- Transportation polytopes
- Flow polytopes

<u>Def</u>: The <u>Birkhoff polytope</u> a.k.a the polytope of doubly stochastic matrices

$$B_n = \left\{ \begin{array}{l} A=(a_{ij}) \text{ real} \\ n\times n \text{ matrices} \end{array} \middle| \begin{array}{l} \cdot\, a_{ij} \geq 0 \\ \cdot \text{ all row \& column sums are } 1 \end{array} \right\}$$

<u>Ex</u>. $B_2 = \left\{ \begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix} \middle| \ 0 \leq a \leq 1 \right\} =$ 

- $\dim B_n =$ 

put anything in here

$\Rightarrow$ unique way to figure out remaining entries

$\Rightarrow \dim = (n-1)^2$

- facets of $B_n = \{ A \in B_n \mid a_{ij} = 0 \}$

For a certain (i,j)

- n^2 facets, for $n \geq 3$

But for $n=2$, setting top left $a=0$ same as setting bottom right equal to 0

- <u>Birkhoff von Neumann Thrm</u>.

Vertices of B_n are exactly permutation matrices.

↳ $n!$ vertices

<u>Def</u>: $P \subset \mathbb{R}^N$ any convex set. Then $p \in P$ is called an <u>extreme point</u> if $\nexists$ non-zero vector $v \in \mathbb{R}^n$ s.t. $[p-v, p+v] \subseteq P$.

Ex.

This specific v works

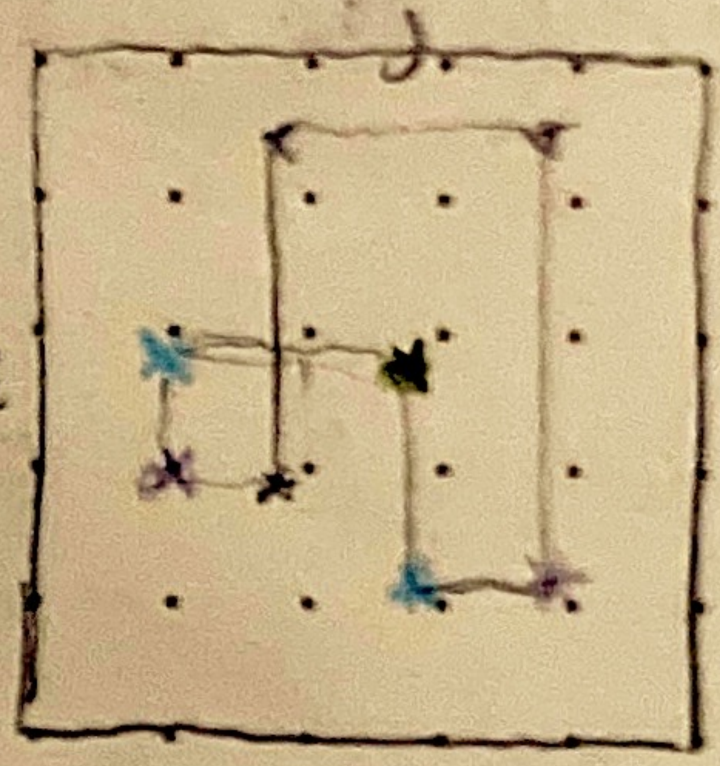
all extremal points

for interior points, any random small enough vector works to get $[p-v, p+v] \subset P$.

Krein-Milman Thm:
$\forall$ compact convex $P \subset \mathbb{R}^n$ $(P = \bar{P})$,
$P = \text{conv}(\text{its extreme pts})$.

Lemma: For a polytope P, extreme pts = vertices of P

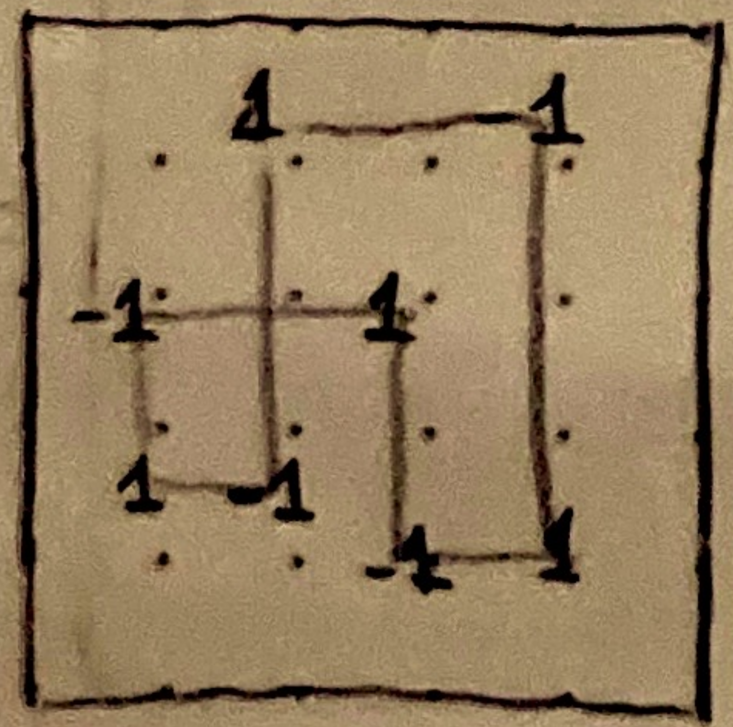
Proof of Birkhov von Neumann Thm:
Let us show any non-permutation matrix is not permutation matrix. Pick non-perm $A \in B_n$.



$\Rightarrow \exists$ entry a_{ij} of A s.t. $0 \lneq a_{ij} \lneq 1$

Can find other non-zero entries in same row & column. And then in same row & column of those, and of those etc.

$\Rightarrow$ Get a cycle of non-zero entries

$B =$ 

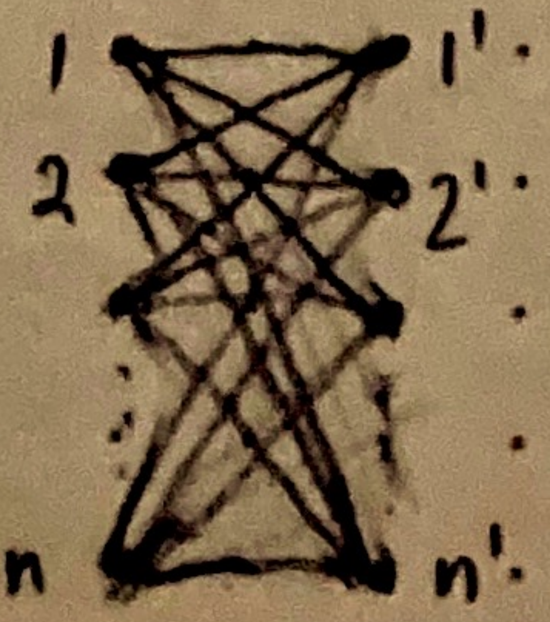
all rows and column sums are 0

and $[A - \varepsilon B, A + \varepsilon B] \in B_n$ for sufficiently small ε $\leq |a_{ij}|$ $\forall a_{ij}$ in the cycle

$\Rightarrow A$ is not a vertex.

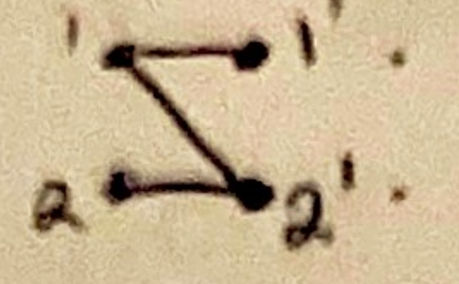
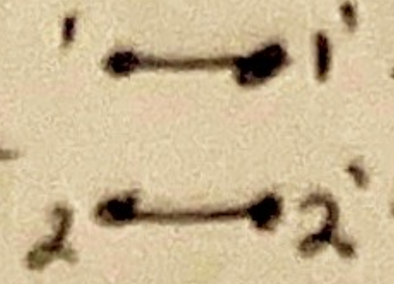
$\Rightarrow$ By symmetry, all perm. matrices are vertices.

Faces of B_n

$G \subseteq K_{n,n} =$ 

$F_G := \{ A \in B_n$ s.t. $a_{ij} = 0$ if (i, j') is not an edge of $G \}$

Ex. $n=2$

$G_1 =$  $G_2 =$ 

$A = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ $\quad A = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$

$F_{G_1} = F_{G_2} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

Q: How do we know which graphs are "good graphs" that are in bijection with faces?

Def: A perfect matching $M \subset G \subset K_{n,n}$ is a subgraph of G with degree of all vertices $= 1$

A graph $G \subset K_{n,n}$ is matching covered if

$G = \bigcup$ all matchings in G.

i.e. if every edge is in some perfect matching.

Thrm: Faces of $B_n \xleftrightarrow{\text{bij}}$ non-empty matching covered subgraphs $G \subset K_{n,n}$.

$F_G \longleftarrow\!\!\shortmid \; G$

Vertices of a face are exactly F_M where M is a perfect matching in G.

Transportation Polytopes

$c_1, \dots, c_m, r_1, \dots, r_m \geq 0$

$\sum c_i = \sum r_j$

$T_{m,n}(c_1, \dots, c_m, r_1, \dots, r_n)$

$= \left\{ \begin{matrix} A = (a_{ij}) \\ m \times n \text{ matrix} \end{matrix} \;\middle|\; \begin{matrix} \cdot \text{ row sums } r_i \\ \cdot \text{ column sums } c_i \\ \cdot\ a_{ij} \geq 0 \ \forall\, i,j \end{matrix} \right\}$

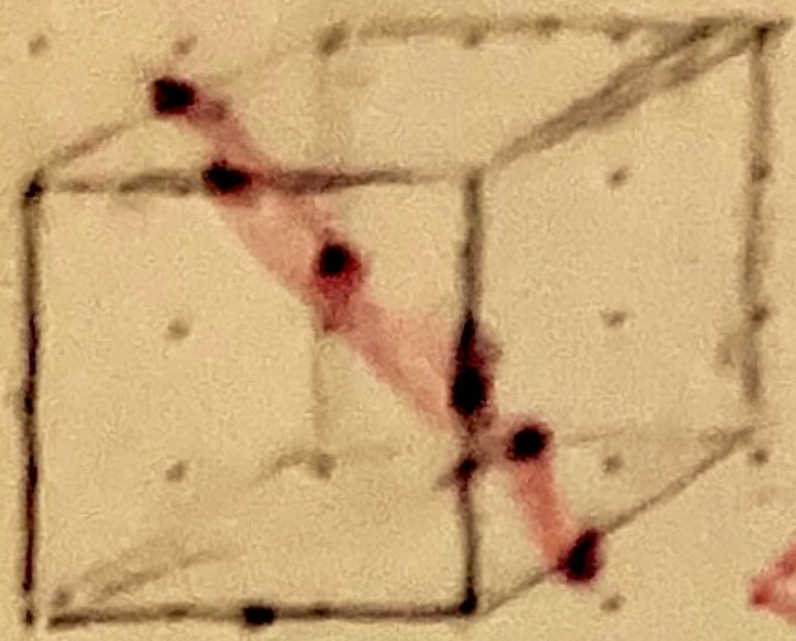
Notation $T_{m,n} := T_{m,n}(n, \dots, n, m, \dots, m)$

$T_{n,n}$ has $n!$ vertices

Ex. $T_{3,2}$ $\quad A = \begin{bmatrix} a & 2-a \\ b & 2-b \\ c & 2-c \end{bmatrix} =$  (see next page for why)

$a+b+c = 3$

$0 \leq a, b, c \leq 2$



$x+y+z=3$

plane through centerpoints, Get a hexagon.

<u>E.x.</u> $T_{n+1,n}$

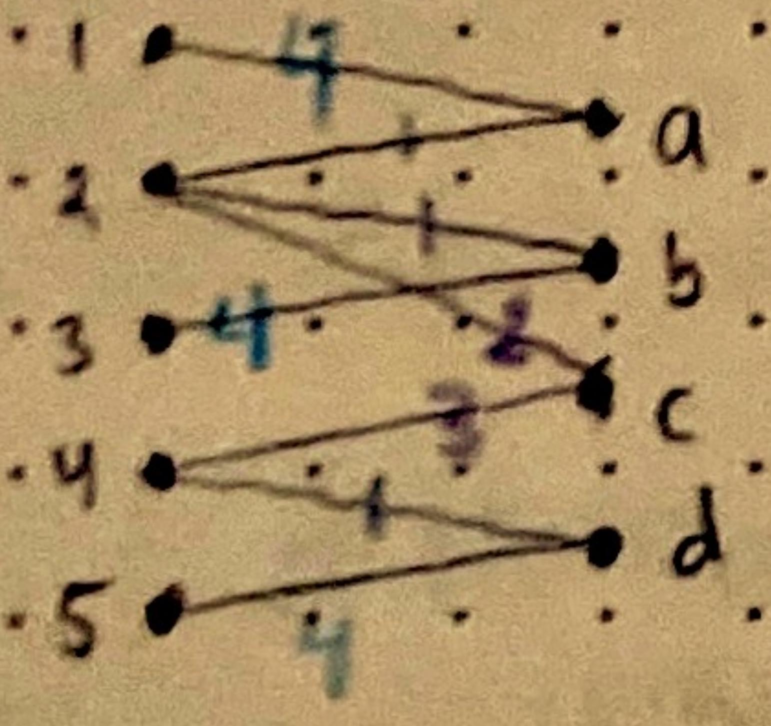
<u>Thrm</u>: $T_{n+1,n}$ has $n!(n+1)^{n-1}$ vertices
(in bijection w/ spanning trees $T \subset K_{n+1}$ with labelled vertices & edges)

Any face of $T_{m,n}(c_1,\dots,c_m,r_1,\dots,r_n)$ corresponds to a certain graph $C \subset K_{m,n}$.

vertices $\xleftrightarrow{\text{bij}}$ some class of forests $\subset K_{m,n}$

For $m=n+1$, "good forests" ~ trees s.t. deg of any vertex in the second part is 2.

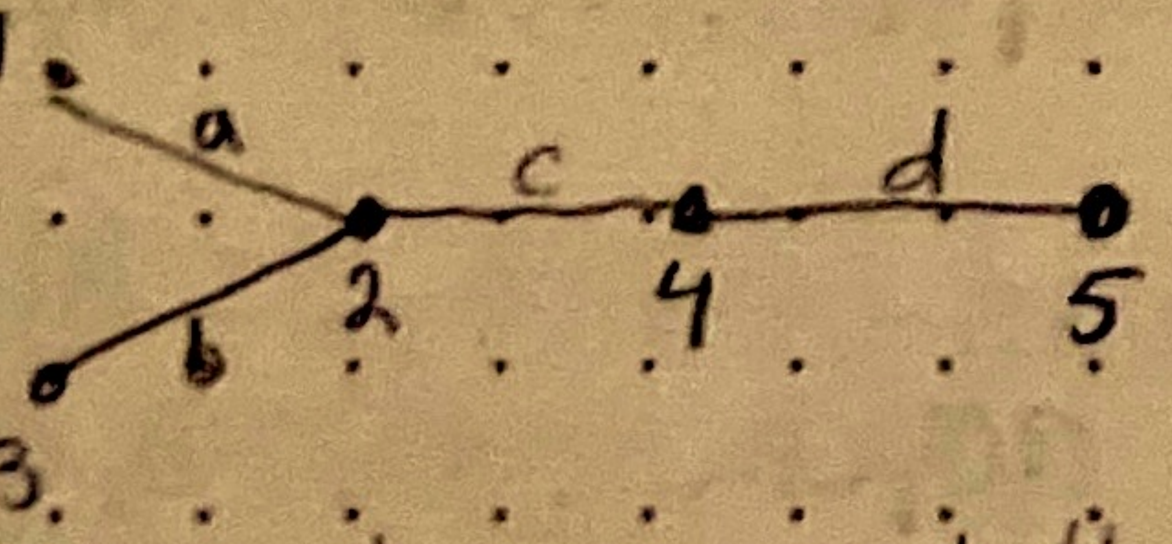
<u>E.x.</u> $T_{5,4}$

$T =$ 

Sum of wts for vertices on <u>left</u> is 4
right 5

Start from leaves and inductively get weight for all edges

$$\begin{array}{c|cccc|} & a & b & c & d \\ \hline 1 & 4 & 0 & 0 & 0 \\ 2 & 1 & 1 & 2 & 0 \\ 3 & 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 3 & 1 \\ 5 & 0 & 0 & 0 & 4 \\ \hline \end{array}$$

$T =$ 

This is how we get the Thrm above.