LECTURE 36 Wed 12/4

Birkhoff Polytope

$$B_n = \left\{ \underset{n\times n}{A = (a_{ij})} \;\middle|\; \begin{array}{l} \cdot\ a_{ij} \geq 0 \\ \cdot \text{ all row sums} = 1 \\ \cdot \text{ all column sums} = 1 \end{array} \right\}$$

$\operatorname{Vol} B_n = ?$

- A combinatorial formula
- Calculated for $n \leq 10$ ⟵ Formula gets computationally huge.

Conj: The Erhat poly. has positive coeff ??
Prof Postnikov is not sure if we should believe this

Chan-Robbins-Yuan Polytope

= the face of B_n given by almost upper-triangular doubly-stochastic matrices

$$A = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & \bullet & \bullet \end{bmatrix}$$

$a_{ij} = 0$ for $i \geq j+2$

Thrm: $\binom{n}{2}! \operatorname{Vol}(CRY_n) = \underbrace{C_1 \cdot C_2 \cdot \ldots \cdot C_{n-2}}_{\text{Catalan \#'s}}$

Conj: CRY
Proved by Zeillberger 1999
One page very analitic proof
(Open Problem?): Find a combinatorial proof

Flow polytopes.

$G = (V, E)$ directed acyclic graph

$V = [n]$. all edges $i \longrightarrow j$ are directed s.t. $i < j$

$c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ s.t. $\sum c_i = 0$

Def: $$\mathrm{Flow}_G(c) := \left\{ (x_e)_{e\in E} \;\middle|\; \begin{array}{l} \cdot\ x_e \geq 0 \\ \cdot\ \forall \text{ vertex } i \in [n] \\ \text{outflow}(i) - \text{inflow}(i) = c_i \end{array} \right\}$$

$\text{outflow}(i) = \sum\limits_{i \xrightarrow{e}} x_e$, $\text{inflow}(i) = \sum\limits_{\xrightarrow{e'} i} x_{e'}$

$\mathrm{Flow}_G := \mathrm{Flow}_G(1, 0, 0, \dots, -1)$

Ex. Flow_{K_3}

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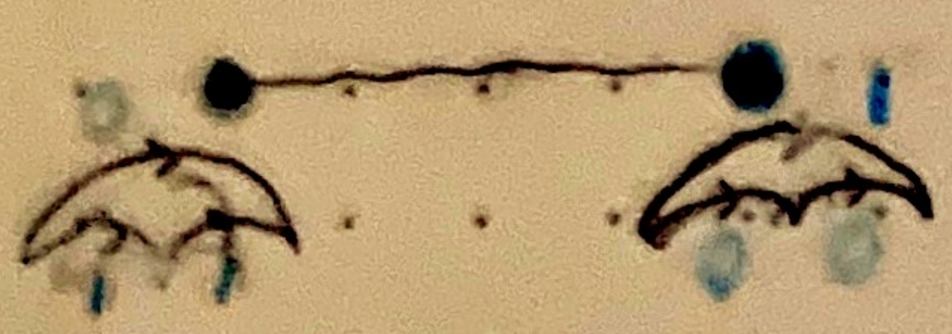
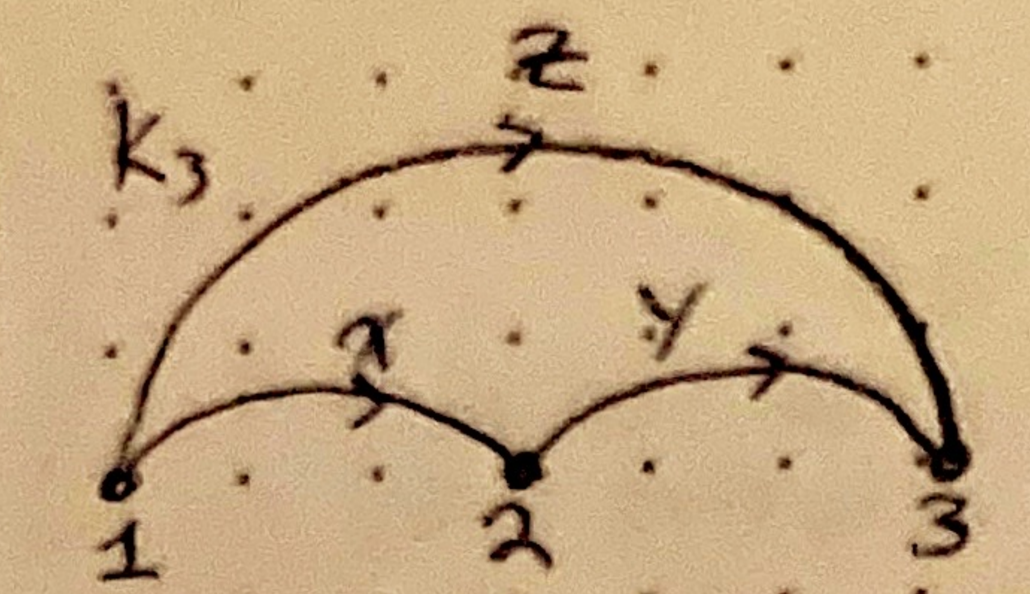
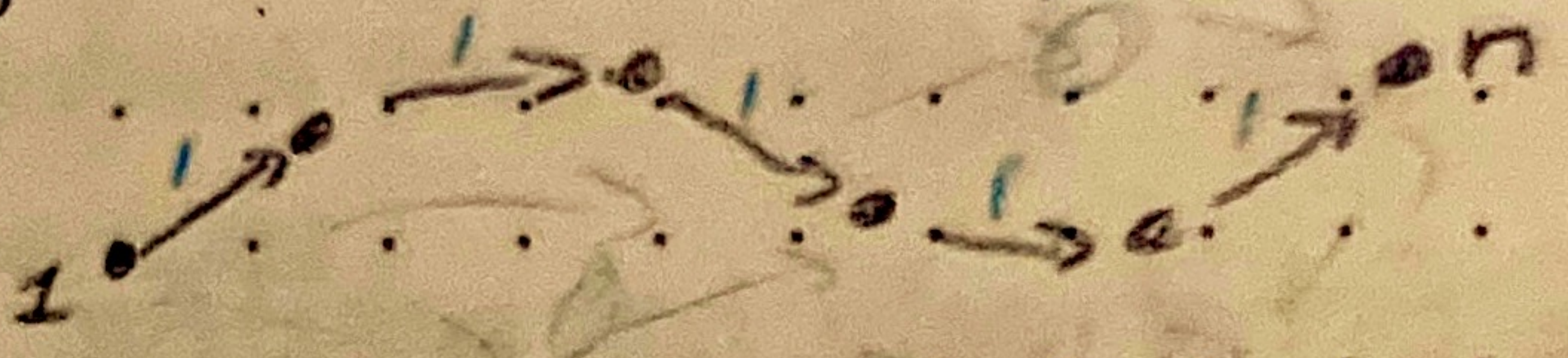




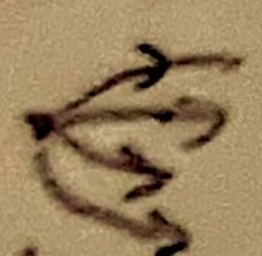
$x, y, z \geq 0$
$x + y = 1$
$x = y$
$-(y+z) = 1$

Prop: Vertices of Flow_G correspond to directed paths from 1 to n in G.



Why:
Want as many zeros as possible & the way to do this is to make one path all 1's and the rest all 0.

$\dim \mathrm{Flow}_G = |E| - |V| + \#$ connected comp. ← connected in the undirected sense

But sometimes it's smaller if we have an internal source  or sink  all these edges have to be zero.

→ True if G has no internal sources or sinks except 1 & n.

↳ Why: $|E|$ vars. Each vertex imposes a relation.

Prop: $CRY_n \cong \mathrm{Flow}_{K_{n+1}}$

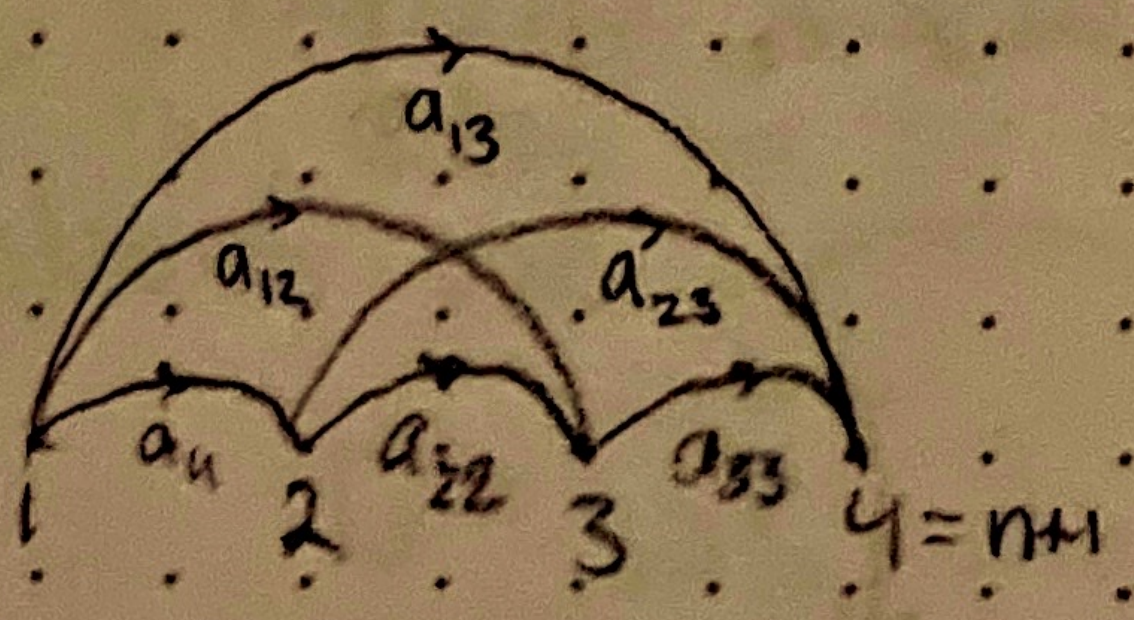
$A = (a_{ij}) \longmapsto (x_{ij})$ $\qquad x_{ij} = a_{i,j-1}$ for $1 \leq i < j \leq n+1$

Ex. $n=3$

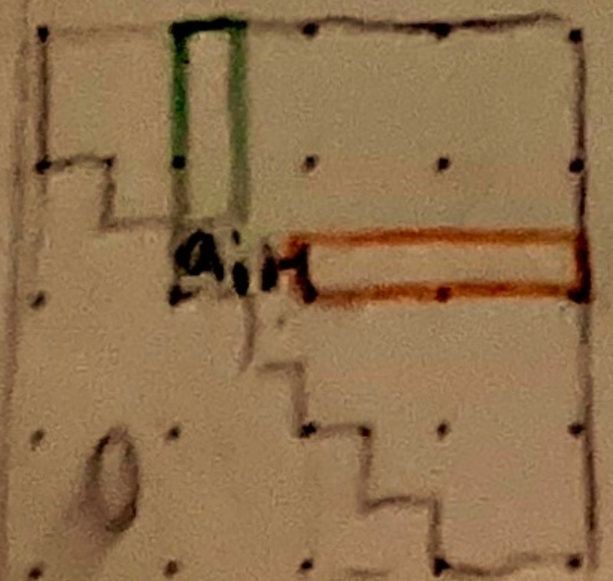
↙ inflow to vert. 2

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} \rightsquigarrow$$

↖ outflow from vert. 2



A =

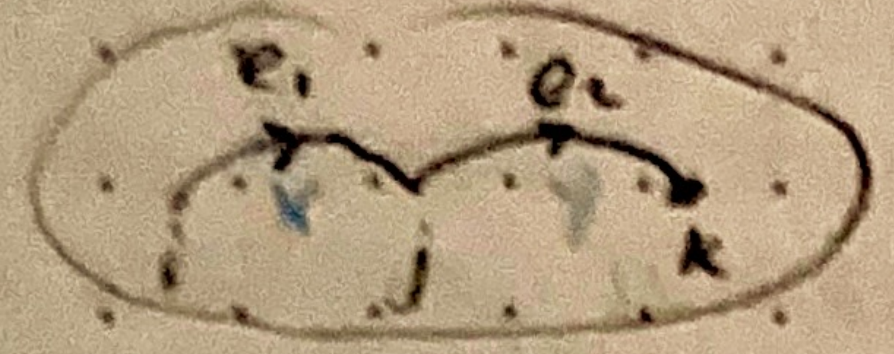


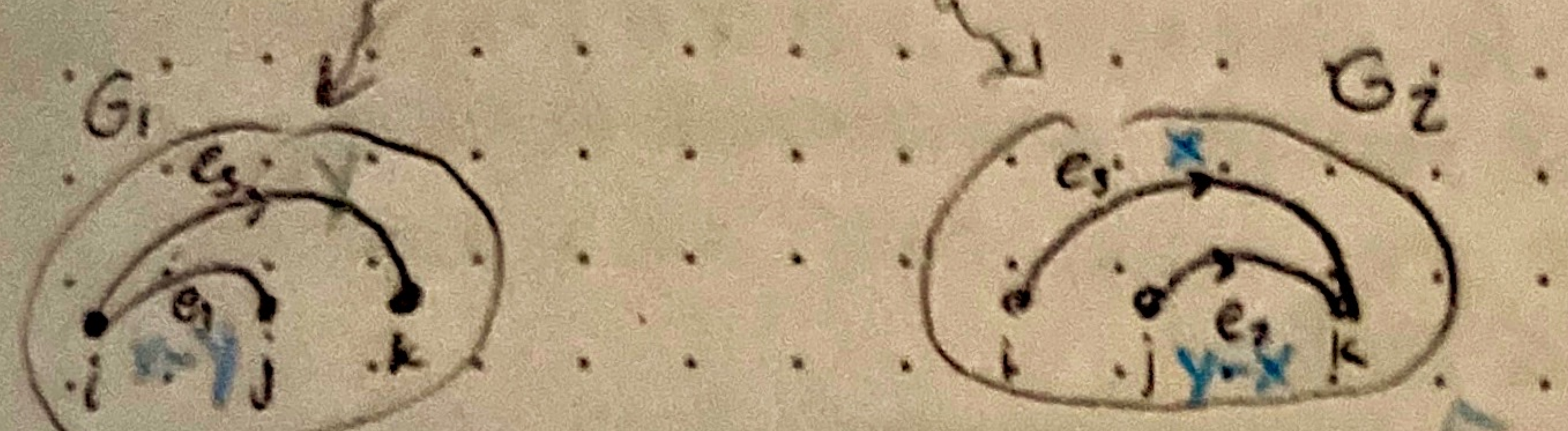
inflow(i) = outflow(i) = $a_{i,j-1}$

Game on graphs

Flows x & y through edges

2 cases
- $x \geq y$
- $x \leq y$

$G =$ 



preserves flow through vertices if $x \geq y$ ... $y \geq x$

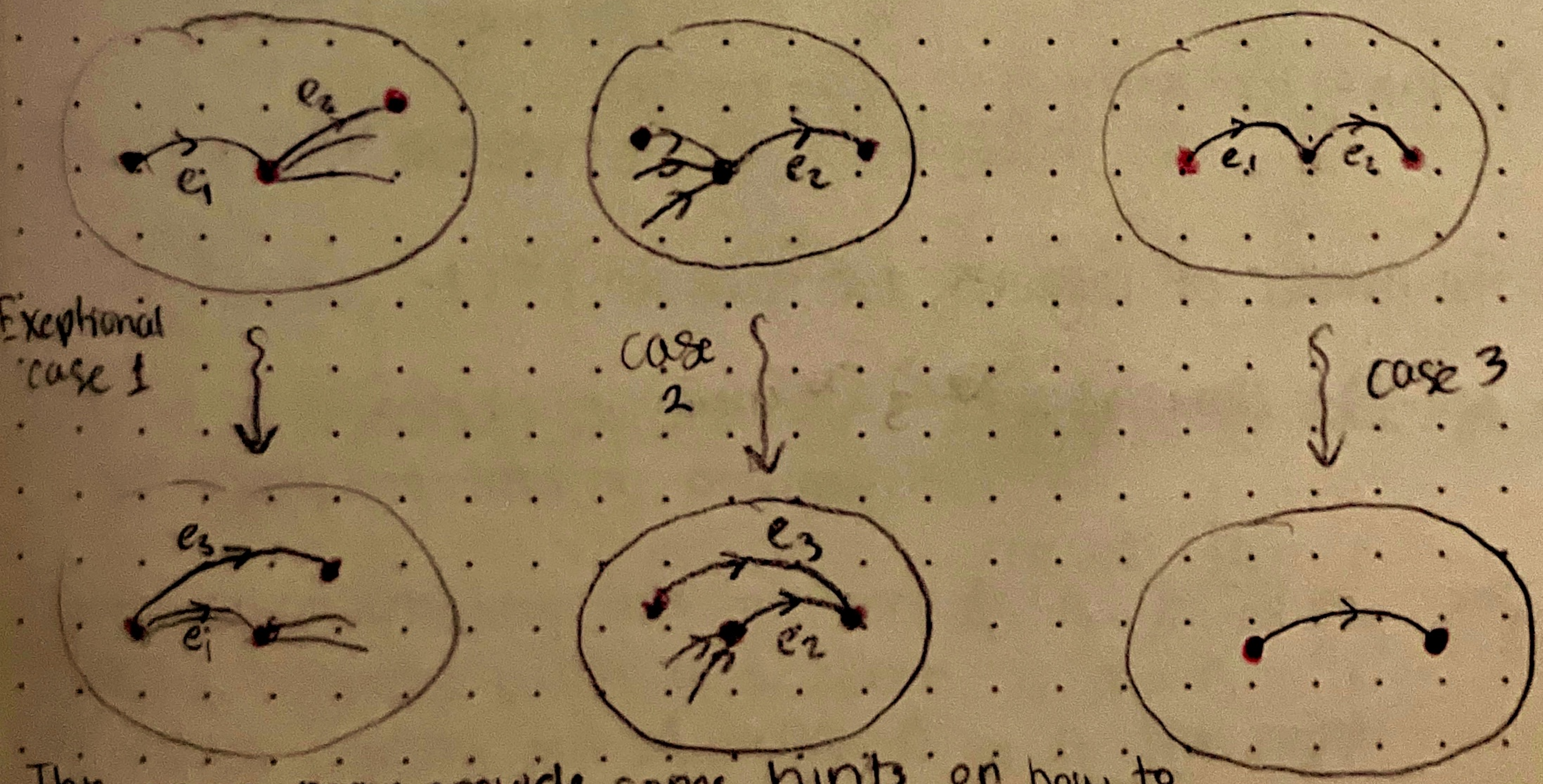
Prop: (1) $\mathrm{Flow}_G = \mathrm{Flow}_{G_1} \cup \mathrm{Flow}_{G_2}$
The intersection is common face given by $x = y$.
(2) $\mathrm{Vol}(\mathrm{Flow}_G)$ "=" $\mathrm{Vol}(\mathrm{Flow}_{G_1}) + \mathrm{Vol}(\mathrm{Flow}_{G_2})$

Not always equality though!

If e_1 is only edge j, for example, then need $y-x=0$ so flow through e_2 is 0 in G_2.

Even worse: If e_1 only edge that enters and e_2 only that exits, we are forced to have $x=y$ in all 3 polytopes, and all three polytopes coincide.

Upshot: Almost same as game described on pset, but this has some exceptional cases



This game may provide some hints on how to solve pset problem version