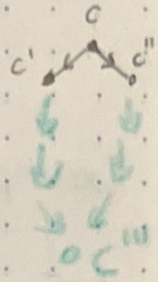
# LECTURE 3 9/8

"Roman" Lemma $\Gamma$ is a directed graph w/out loops (possibly infinite) s.t.

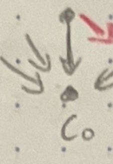
(1) $\Gamma$ is connected as an undirected graph

(2) $\forall$ forks (c; c', c'') in $\Gamma$, $\exists$ $c''' \in \Gamma$ and 2 directed paths of the same length from $c'$ to $c'''$ and $c''$ to $c'''$.



Then either graph g has no sinks (a sink: $\to \cdot \leftarrow$)
OR it has exactly one sink $c_0$ and any directed walk on $c_0$ eventually ends at $c_0$

**Proof**: Base start with the sink



impossible

$c_0$

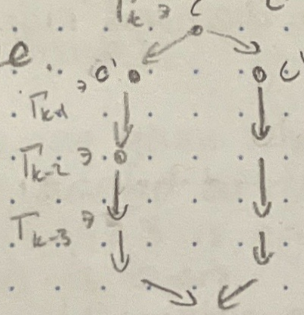
(no edges to $c_0$ can have a different edge down from it)

Let $\Gamma_k$ = the set of vertices $c \in \Gamma$ s.t. $\exists$ directed path of length $k$ from $c$ to $c_0$ and no shorter directed path
e.g. $\Gamma_0 = c_0$; $\Gamma_1$ = all vertices adjacent to $c$

**Claim**: $\forall\, c \in \Gamma_k$, any directed edge from $c$ goes to $\Gamma_{k-1}$.

**Proof**: induction on $k$.

Let $c \in \Gamma_k$, $\exists$ a directed edge $c \to c'$ where $c' \in \Gamma_{k-1}$. let $c \to c''$ be any other edge.



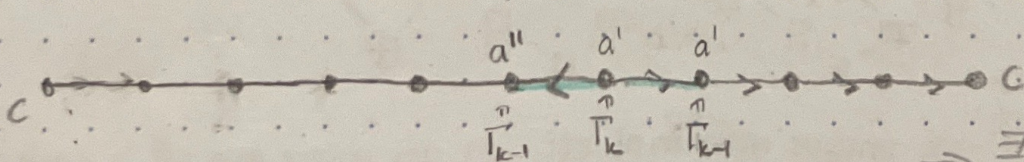
$\Gamma_k \ni c$, $\Gamma_{k-1} \ni c'$, $c''$, $\Gamma_{k-2} \ni$, $\Gamma_{k-3} \ni$

Now We need to prove that $\bigcup_k \Gamma_k = \Gamma_k$.
i.e. that everything ends up at the sink eventually.

Let $c$ be any vertex of $\Gamma$

Pick a shortest (undirected) path P between $c$ and $c_0$ with out of order edge

P.



$a''$ $a'$ $a'$

$c$ $c_0$

$\Gamma_{k-1}$ $\Gamma_k$ $\Gamma_{k-1}$

b/c this is a fork

$\Rightarrow$ $\exists$ shorter path from $a''$ to $c_0$.
$\Rightarrow$ contradiction

$\Gamma$ should be the Hasse diagram of a ranked poset with a unique minimal element.

$\Gamma$

Examples of such posets

**Ex. 1** <u>Weak Bruhat order $B_n$</u>

Elts are permutations $w = w_1 \dots w_n$ of $1, 2, \dots, n$

Covering relations are <u>adjacent transpositions</u>

$$w = w_1 \cdots w_i w_{i+1} \cdots w_n \longrightarrow w' = w_1 \cdots w_{i+1} w_i \cdots w_n$$

where $w_i > w_{i+1}$

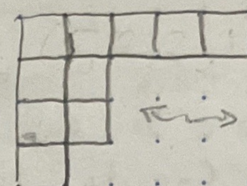
**Ex. 2** <u>Young's Lattice $\mathbb{Y}$</u>

Elements: partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell \geq 0$

Graphically represented by Young diagrams

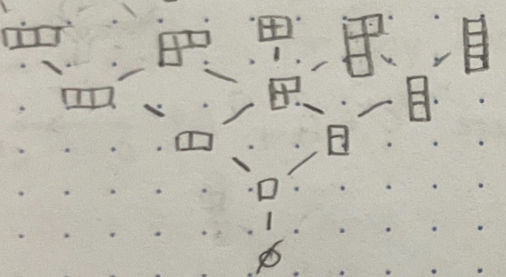
e.g.



$\longleftrightarrow (5, 2, 2, 1)$

Covering relation $\longrightarrow$ removing a single box from Young diagram.

$\mathbb{Y}$



$\leftarrow$ This one actually satisfies the diamond rule

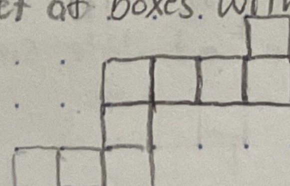
**Ex. 3** $\mathbb{Y}^{(n)}$ (for a fixed $n$)

Elements: all partitions

Covering relations $\longleftrightarrow$ Removing $n$-ribbons

<u>Def</u>: $n$-ribbon is a connected set of boxes with at most 1 box in each diagonal

e.g.

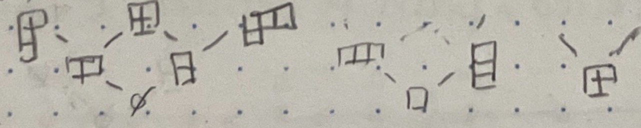


$\leftarrow$ A 9-ribbon

$\mathbb{Y}^{(1)} = \mathbb{Y}$

A "diamond"

$\mathbb{Y}^{(2)}$



$\uparrow$ no unique minimal elt, but yes unique min. elt in each <u>connected component</u>

**Def**: The $n$-core of $\lambda$ is the partition obtained from $\lambda$ by erasing $n$-ribbons until we can no longer erase them.

**Exercise** (will be on pset 1):
Show that the $n$-core of $\lambda$ is well-defined
(i.e. the result does not depend on the order we remove strips).

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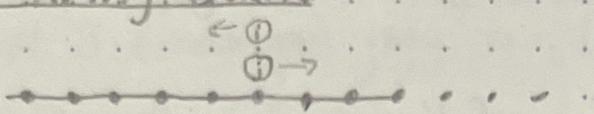
## Labeled chip-firing Game:

$G =$

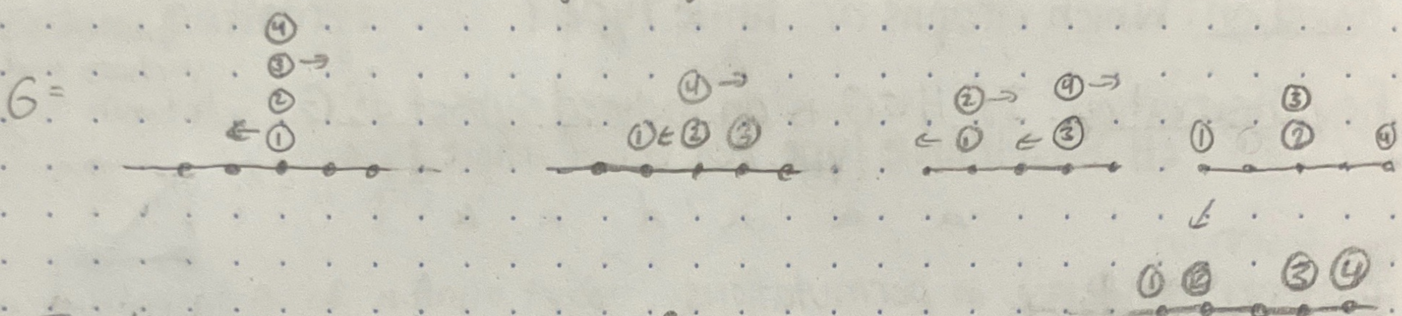


infinite chain

each chip is now labelled.

**Moves**: Can move chips (i) to left and (j) to right from same vertex if $i < j$

e.x. $G =$



**Question**: Is this game always confluent?

**Ans**: NO

E.x.

**Thrm** (Hopkins, McConville, Propp):

The labelled chip-firing game where the initial configuration consists of $n$ chips located at the same vertex
- IS NOT confluent if $n \geq 3$ is odd.
- IS confluent if $n$ is even.

This is harder because Roman Lemma does not hold

Widely open problem: How to classify confluence for more general initial configurations.