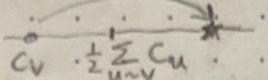
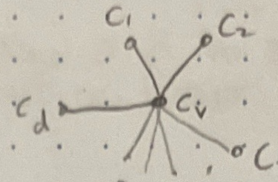
LECTURE 4 9/10/25

Recall:

Kostant's Game:

- Chip configurations on graph G
- Start with one chip at one vertex
- Moves: Reflections





$C_v \rightsquigarrow C_v' = C_1 + \dots + C_d - C_v$
if $C_v < \frac{1}{2}(C_1 + \dots + C_d)$

Play this game until all vertices become happy (i.e. $C_v \geq \frac{1}{2}\sum_{u\sim v} C_u$ for all v), or play forever

Def: A graph G is of finite type if Konstants game stops

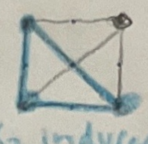
(proved last lecture this is a well defined property no matter how we play the game)

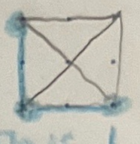
Question: Which graphs are finite type?

Key Observation: If $H \subset G$ is an induced subset of G & H is of infinite type, then G is of infinite type.

← Given by a subset of vertices and all edges between these vertices.

Ex:


K_3 induced subgraph of K_4


This is NOT

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Digression on **Pattern Avoidence** in permutations

Def: For a permutation $\pi = \pi_1, \dots, \pi_k$, we say a larger permutation $w = w_1, \dots, w_n$ contains pattern π if $\exists$ subword $w_{i_1}, w_{i_2}, \dots, w_{i_k}$ with $i_1 < i_2 < \dots < i_k$ s.t. these entries are in the same relative order as entries of π. Otherwise say w avoids pattern π or is π-avoiding.

Ex. $w = 53\overset{3}{7}2\overset{2}{6}1\overset{1}{4}$ ← contains pattern 321
But it is 123 avoiding

NOTE: All permutations in one-line notation

Thrm: $\forall$ patterns π of size 3, # π-avoiding perms of size n
$= C_n = \frac{1}{n+1}\binom{2n}{n}$, the Catalan #

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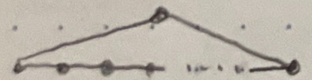
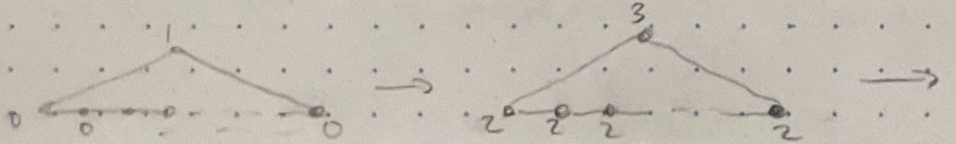
Can we apply the same pattern avoidence idea, but on graphs? Find some forbidden patterns (induced subgraphs)

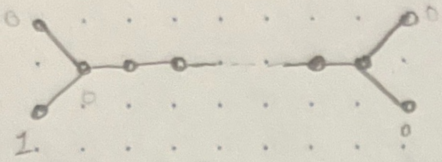
1)  cycles (we saw this last lecture)



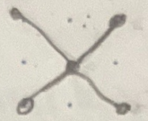
etc. never ends.

⇒ Any connected G of finite type must be a tree

2.) Another forbidden subgraph

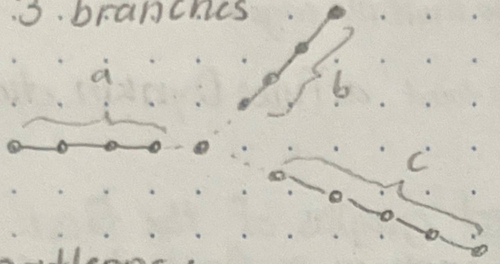


SPECIAL CASE :



⇒ Any G of finite type has at most 1 branching point, with at most 3 branches

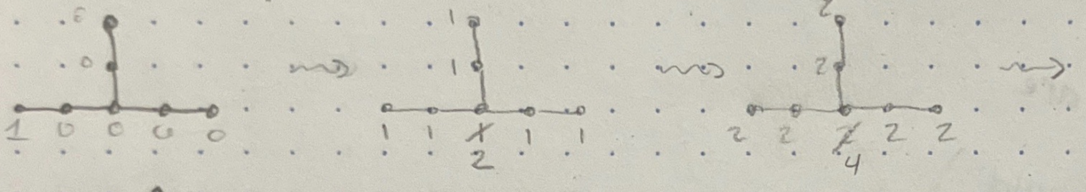
i.e. G =



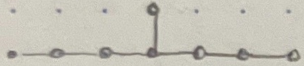
$a \leq b \leq c$

More forbidden patterns :

3.)



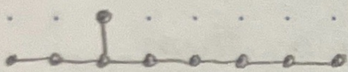
4.)



is of infinite type.

⇒ if a=1, then $b \leq 2$

5.

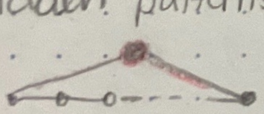


⇒ if a=1 and b=2, then $c \leq 4$

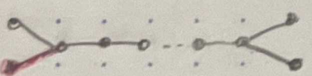
Claim: This is the complete list of forbidden patterns :

Forbidden patterns (of infinite type)

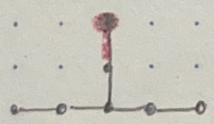
$\tilde{A}_n$



$\tilde{D}_n$

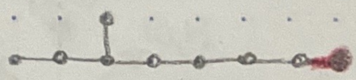


$\tilde{E}_6$



$\tilde{E}_7$

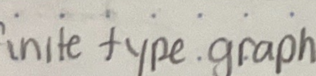
$\tilde{E}_8$



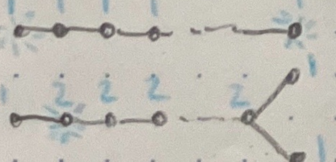
n = # vertices − 1

All finite type graphs

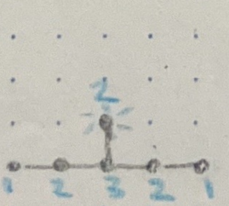
A_n



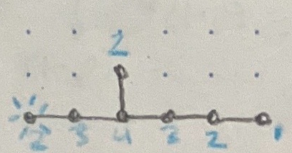
D_n



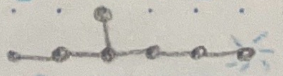
E_6



E_7



E_8



n = # vertices

← exercise

Anything not containing induced subgraph on 1st list is in 2nd list

Exercise: Check all 2nd list graphs are finite type. (final config. shown in blue)

Observations: There is a 1-1 correspondence between finite-type graphs and THE minimal list of forbidden patterns.

↑ It is canonical!

These graphs have names.

A_n, D_n, E_6, E_7, E_8 : Simple, simply-laced Dynkin diagrams of finite type

↑ connected ↑ no multiple edges

$\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ Simple, simply-laced affine Dynkin diagrams

Note the second ones are the extended graphs of the first.
(connect a new vertex to the excited vertices in final chip configuration)

Overall, there are

(finite type } "wild" type)

HEAVEN affine type HELL

purgatory

Somehow we've now classified root systems without defining them....

Def: (Root system from this perspective, NOT standard def)

Root system = the set of all chip configurations you could wind up at at some point while playing Kostant's game of a Dynkin diagram of finite type & their inverses.

Ex. A_2