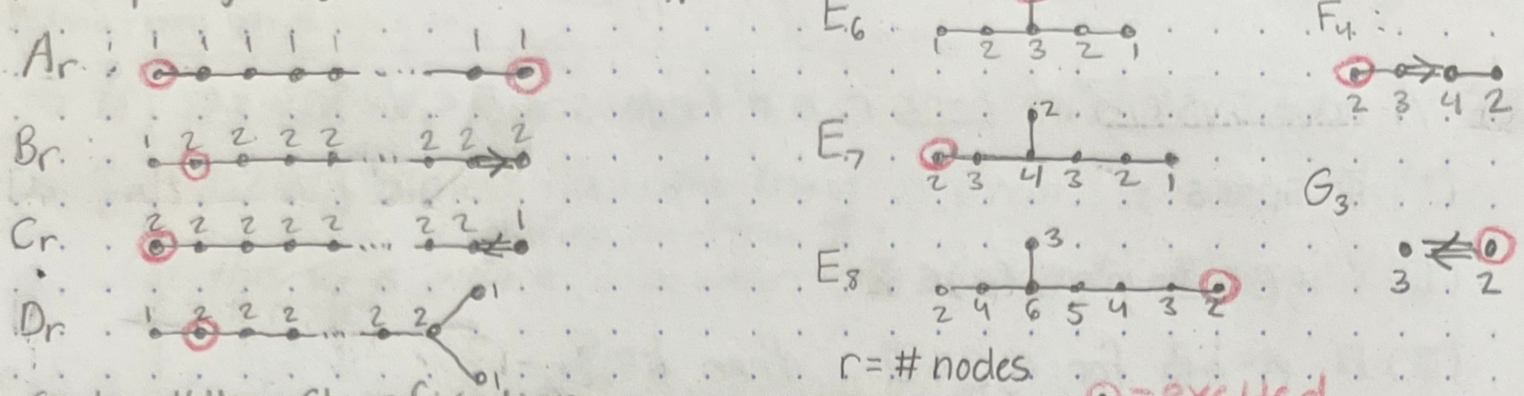
LECTURE 5 Fri 12

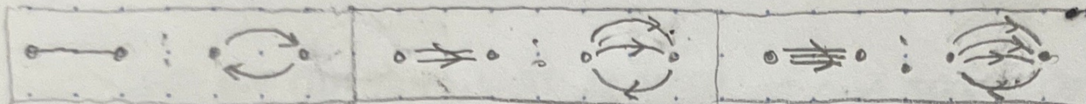
ALL Dynking diagrams of finite type



r = # nodes. ⊙ = excited

Cartan Killing Classification

What do multiple edges mean?



Count # of chips of the neighbor once for each directed edge in that direction.

Q: How does this relate to regular definition of root systems?

Why do we have v happy $\rightsquigarrow$ $c_v \geq \frac{1}{2}\sum_{u \sim v} c_u$?

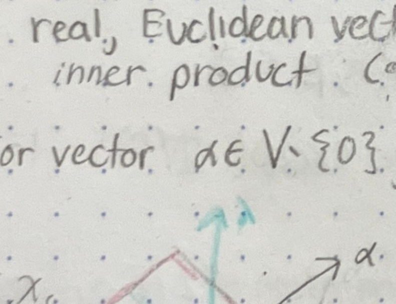
Why not 1 or the average?

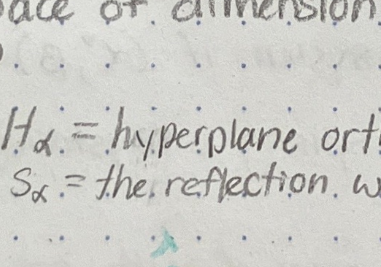
Standard Definition of Root Systems:

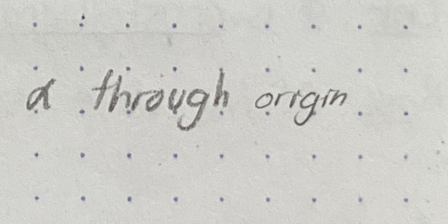
V real, Euclidean vector space of dimension r
inner product $(\cdot, \cdot)$

For vector $\alpha \in V \setminus \{0\}$, let H_α = hyperplane orthogonal to α through origin
S_α = the reflection w.r.t. H_α

Ex.







In equations, $S_\alpha : \lambda \longmapsto 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}\alpha$

Sanity check $S_\alpha(\alpha) = \alpha - 2\frac{(\alpha,\alpha)}{(\alpha,\alpha)}\alpha = -\alpha$ ✓

Def: For a <u>root</u> α, the <u>coroot</u> $\alpha^\vee = \frac{2}{(\alpha,\alpha)} \cdot \alpha$.

Then we can write $s_\alpha : \lambda \mapsto \lambda - (\alpha^\vee, \lambda)\alpha$

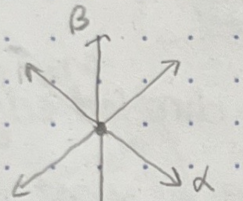
Pretty sure $\langle \alpha, \lambda \rangle = (\alpha^\vee, \lambda)$ in a lot of rep theory books.

> **Def**: A <u>root system</u> of <u>rank</u> r is a finite set $\Phi \subset V \setminus \{0\}$ s.t.
>
> (I) Φ spans V.
>
> (II) $\forall\, \alpha, \beta \in \Phi$, $s_\alpha(\beta) \in \Phi$
>
> (III) If $\alpha = k\beta$ for $\alpha, \beta \in \Phi$, then $k = \{1, -1\}$

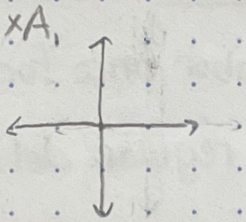
<u>Ex.</u> $r=1$ $A_1 = -\alpha \longleftarrow\!\bullet\!\longrightarrow \alpha$

$r=2$

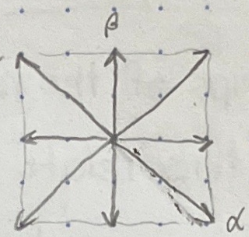
A_2



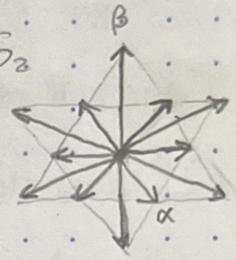
$A_1 \times A_1$



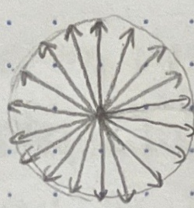
B_2



G_2



$I^{(n)}$ 2^n vectors



Def: Φ is <u>crystallographic</u> root system if $(\alpha^\vee, \beta) \in \mathbb{Z}$ $\forall\, \alpha, \beta \in \Phi$.

<u>Rmk</u>: Of the above examples, everything is crystallographic <u>except</u> $I^{(n)}$.

Root systems appear in many areas of math, ex. representation of Lie groups.

Def: A (finite) <u>reflection group</u> W is a finite subgroup of $GL(V)$ generated by some reflections s_α for some vectors $\alpha \neq \vec{0}$.

Def: A (one) Weyl group is a finite subgroup of $GL(V)$ generated by all s_α for α in a given crystallographic root system.

$$\left\{\begin{matrix}\text{Weyl}\\ \text{Groups}\end{matrix}\right\} \subset \left\{\begin{matrix}\text{reflection}\\ \text{groups}\end{matrix}\right\} \subset \left\{\begin{matrix}\text{Coxeter}\\ \text{groups}\end{matrix}\right\}$$

$s_\alpha^2 = id$

<u>Observation</u>: If $s_\alpha, s_\beta \in W$, then

$$\underset{= s_\alpha^{-1}}{s_\alpha} s_\beta s_\alpha = s_{s_\alpha(\beta)}$$

$\Rightarrow$ If $s_\alpha, s_\beta \in W$, then $s_{\alpha(\beta)} \in W$

$\Rightarrow$ $\{\alpha \mid s_\alpha \in W\}$ "almost the same thing" as a root system (satisfies condition II).

But this set is infinite (and doesn't care about lengths of vectors)

$\{\text{Finite reflection groups}\} \xleftrightarrow{\text{bij}} \{\text{root systems}\}/\text{rescaling of vectors}$

Ex. B_r & C_r different root systems for $r > 2$, but equivalent up to rescaling of vectors

Somehow, we can start with a basis of our space and generate the rest of the root system via reflections.
But the basis has to be particularly nice in certain ways we will get to next lecture.