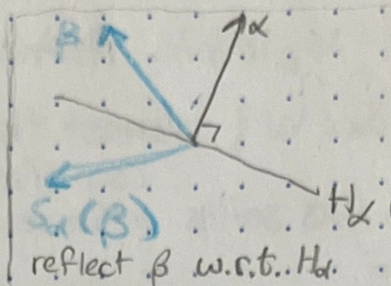
LECTURE 6 Mon 9/15

Recall

Root Systems: $\Phi \subset V \setminus \{0\}$ $\quad V \simeq \mathbb{R}^r$

$(\cdot,\cdot)$ inner product

0. Φ spans V
1. $\forall\, \alpha, \beta \in \Phi$, $s_\alpha(\beta) \in \Phi$
2. If $\alpha = k\beta$ for $\alpha, \beta \in \Phi$ then $k \in \{\pm 1\}$



Reflections: $s_\alpha : \lambda \mapsto (\lambda, \alpha^\vee)\alpha$

where $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ ← This is the coroot

Φ is crystallographic if

3. $c_{\alpha\beta} := (\alpha^\vee, \beta) \in \mathbb{Z}$ $\quad \forall\, \alpha, \beta \in \Phi$ ← $c_{\alpha\beta}$ are called Cartan integers

FACT: The dual root system of Φ is $\Phi^\vee$, consisting of all the dual roots.

$W = W_\Phi$ group generated by s_α for $\alpha \in \Phi$

↑ Reflection group

We also use the word Weyl group when Φ is crystallographic.

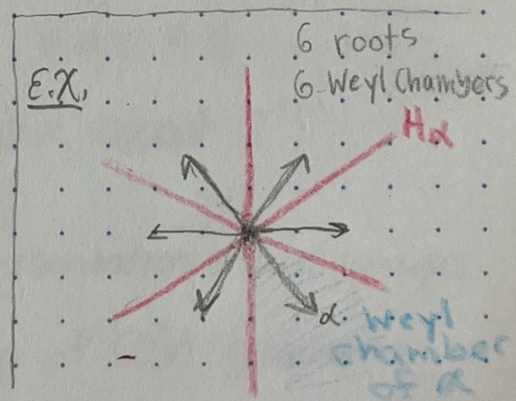
How can we pick a nice basis for V of roots in Φ?

First, we will want...

Def: Coxeter arrangement $\mathcal{A} = \mathcal{A}_\Phi$

= the set of hyperplanes H_α, $\alpha \in \Phi$ in V.

Regions of $\mathcal{A}$ are called Weyl Chambers



Ex. Root systems of type A_{n-1} $\quad (r = n-1)$

$\Phi = \{\vec{e}_i - \vec{e}_j \mid i \neq j \in [n]\} \subset V = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \ldots + x_n = 0\}$

where $\vec{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ ← i^{th} spot

Roots: $\alpha_{ij} = \vec{e}_i - \vec{e}_j$

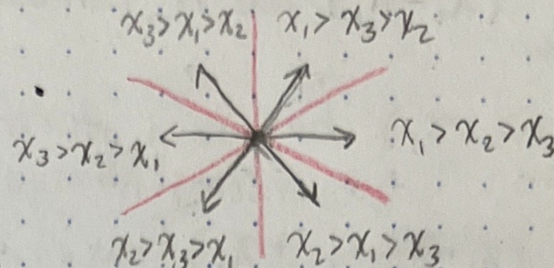
Hyperplanes: $H_{ij} = H_{ji} = \{(x_1, \ldots, x_n) \mid x_i = x_j\}$

Reflections: $s_{ij} : (x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) \mapsto (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)$

$W = S_n$ symmetric group of permutations w acting on V by permutations of coordinates

roots $= n(n-1)$

Weyl chambers $= n!$ given by linear order on coordinates



[Combinatorics does a lot of work it type A

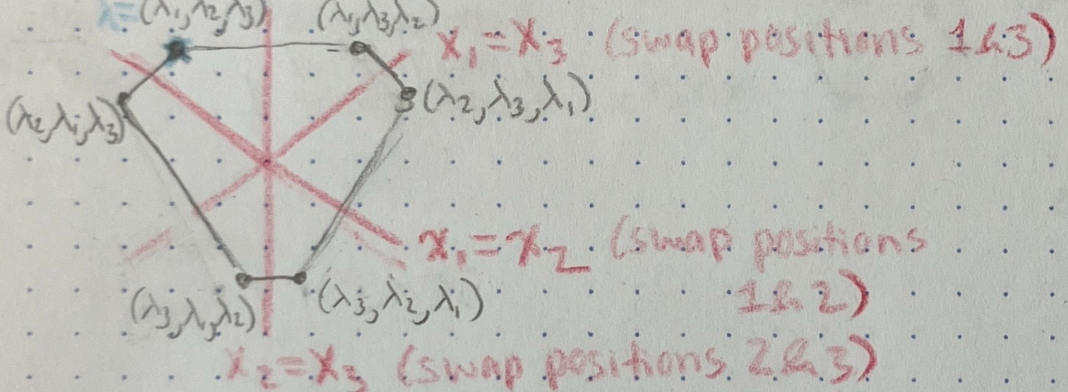
Lie Theorists think type A is a "baby case" and want to prove for all type

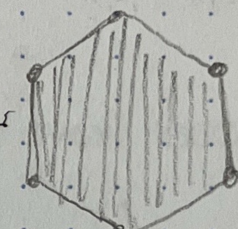
↳ E_8 is most complicated, so if you can show that hooray

Def: <u>Permutohydron</u>

$$P_n(\lambda) = P_n(\lambda_1, \dots, \lambda_n) := \text{Convex Hull}\,((\underbrace{\lambda_{w_1}, \lambda_{w_2}, \dots, \lambda_{w_n}}_{= w^{-1}(\lambda)}) \mid w \in S_n)$$

Ex. $P(\lambda_1, \lambda_2, \lambda_3) =$



$P_2 =$ 

In general, for $\lambda_1 > \lambda_2 > \dots > \lambda_n$,

Vertices of $P_n(\lambda) \xleftrightarrow{\text{bij}} \{\text{Weyl Chambers}\} \xleftrightarrow{\text{bij}} W$

Def: The standard permutohedron $P_n = P_n((n-1, n-2, n-3, \dots, 1, 0))$

vertices $\longleftrightarrow$ Weyl chambers , facets (top dim'l faces) $\longleftrightarrow$ Roots

Thrm: P_n = Newton Polytope of $\prod_{1 \le i < j \le n} (x_j - x_i) = \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}$

Def: Netwon Polytope $(f(x_1, \dots, x_n))$

$= \text{Convex Hull}\,((a_1, \dots, a_n) \mid x_1^{a_1} \cdots x_n^{a_n}$ is non-zero monomial of $f(x))$

Facts

① $\prod_{1\le i<j\le n} (x_i - x_j)$ has $n!$ non-zero terms

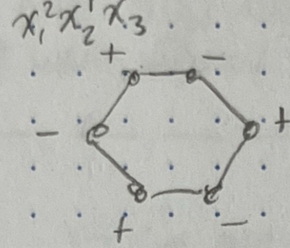
② $\prod_{1\le i<j\le n} (x_i + x_j)$ has $\underline{\qquad}$ terms # forest on n labeled vertices
= # lattice points of permutohedron

②' $\mathrm{Vol}(P^n) = n^{n-2} =$ (# labeled trees on n vertices)

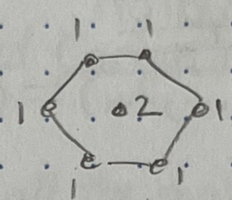
③ $\prod_{1\le i<j\le n} (x_i + x_{i+1} + \dots + x_j)$ has Catalan #, terms with coefficient 1.
$\underline{\qquad}$ terms total

Ex. $\underline{n=3}$

① $(x_1 - x_2)(x_2 - x_3)(x_1 - x_3)$



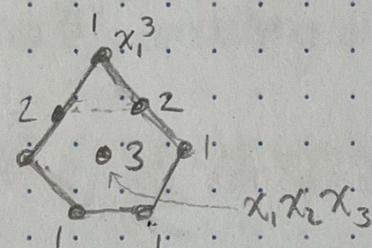
6 monomials



(labels are the coefficient)

② $(x_1 + x_2)(x_2 + x_3)(x_1 + x_3)$ 7 monomials

③ $(x_1 + x_2)(x_2 + x_3)(x_1 + x_2 + x_3)$



8 monomials
5 terms w/ coefficient 1

Exercise: Prove these facts.