LECTURE 7 Wed 9/17

Φ root system,

- s_α reflections w.r.t. hyperplanes $H_\alpha \perp \alpha$
- $\mathcal{A} = \mathcal{A}_\Phi = \{H_\Phi\}$ <u>coxeter arrangement</u>, regions of $\mathcal{A}$ are <u>Weyl chambers</u>
- $P_W(\lambda) = \text{Conv Hull}\{w(\lambda) \mid w \in W\}$ W-permutohedron
  acts on positions

Type A_{n-1}: $\Phi = \{\vec{e}_i - \vec{e}_j \mid i \neq j \in [n]\}$

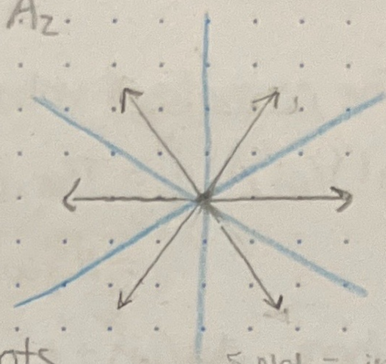
$W = S_n$ symmetric group

$\mathcal{A} = \{x_i - x_j\}$ braid arrangement

$s_{e_i - e_j} = (i,j) \in S_n$ transpositions
swap values in positions i & j

$P_n(\lambda) = P_{S_n}(\lambda)$ the permutohedron

A_2



6 roots. Not = in general.
6 Weyl chambers always =
6 Elts of W

The standard permutohedron $P_n = P_n((n-1, n-2, \dots, 1, 0))$.

$\text{Vol}(P_n) = n^{n-2}$ (# labeled trees on n vertices).

But this doesn't live in $V = \{(x_1, x_2, \dots, x_n) \mid x_1 + \dots + x_n = 0\} \supset \Phi$

It lives in the dual space $V^* \cong \mathbb{R}^n / (1, 1, \dots, 1)\mathbb{R}$

We can identify V and V^* because the space is Euclidean

Lives in affine hyperplane because sum of all coords. is constant.
Just shift by $c(1, 1, \dots, 1)$

$P_n(n-1, n-2, \dots, 1, 0) \simeq P_n((n-1, n-2, \dots, 1, 0) - c(1, 1, \dots, 1))$ where $c = \frac{n-1}{2}$.

How do we define volume when space is not top dim'l?

Easiest way: Projection $\rho: \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$

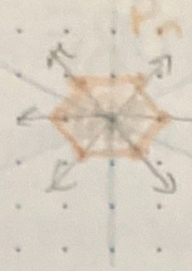
$\rho: (x_1, \dots, x_n) \mapsto (x_1, x_2, \dots, x_{n-1})$

<u>def</u>: $\text{Vol}(P) = \text{Vol}_{n-1}(\rho(P))$

The usual Euclidean $(n-1)$-dim volume in $\mathbb{R}^{n-1}$

Exercise: Check it doesn't really matter which projection you take.

Ex In A_2, $P_n = P_n((2,1,0))$ $c = \frac{1}{2}$



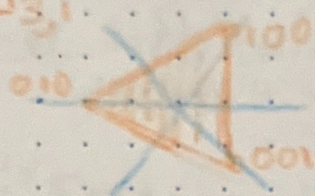
$\Rightarrow P_n \rightsquigarrow P_n(\frac{1}{2}, 0, -\frac{1}{2})$

Vertices are at $\frac{1}{2}\alpha$ for each $\alpha \in \Phi$

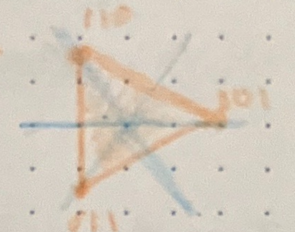
Hypersymplex

$\Delta_{n,k} = P_n((\underbrace{1,\dots,1}_{k}, \underbrace{0,\dots,0}_{n-k}))$

$\Delta_{3,1}$



$\Delta_{3,2}$



Thrm: The normalized volume

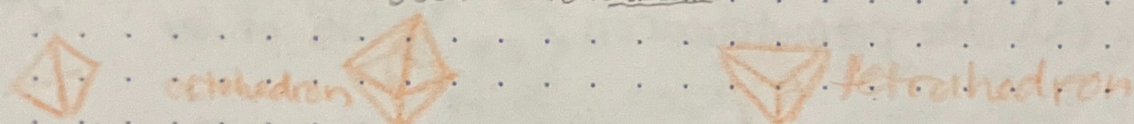
$(n-1)! \operatorname{Vol} \Delta_{n,k}$ = The Eularian number $A_{n-1,k-1}$

$= \#\{w \in S_{n-1} \mid des(w) = k-1\}$

Where $des(w) = \#\{i \mid w_i > w_{i+1}\}$ = # descents in w

such i are descents.

Ex. $n=4$



$3! \operatorname{Vol} \Delta_{4,1} = 1$, $3! \operatorname{Vol} \Delta_{4,2} = 4$, $3! \operatorname{Vol} \Delta_{4,3} = 1$

once you know the first and last are 1, the middle must be 4 so the sum is $|S_3| = 3! = 6$

In general

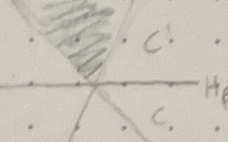
$\{\text{Weyl Chambers}\} \xleftrightarrow{?} W$ (Weyl group)

Pick one Weyl chamber and call it the fundamental Weyl Chamber C_0

Lemma: $W \xleftrightarrow{\text{bij}}$ Weyl Chambers

$w \longmapsto w(C_0)$, $w \in W$

Proof:

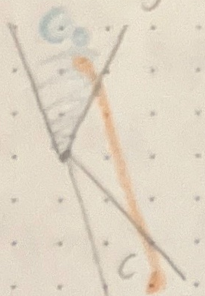


We can easily go from C_0 to all neighboring chambers by reflecting over its bounding hyperplanes

Then keep going reflecting over new bounding hyperplanes

$C' = S_\alpha(C_0)$ $C = S_\beta(S_\alpha) C_0$

How to go from C_0 to a chamber C?



Pick generic points in C & C_0 so that line between crosses through hyperplanes (but not their intersections)

Do reflections corresponding to planes the line hits.

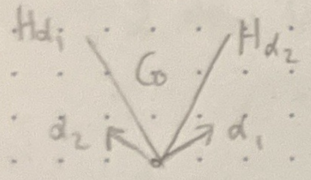
<u>Exercise</u>: Prove injectivity: If $w(C_0) = C_0$, then $w = id \in W$

<u>Ex.</u> Type A_{n-1}, $C_0 = \{x_1 \geq x_2 \geq \dots \geq x_n\}$

Find the minimal collection of roots $\alpha_1, \alpha_2, \dots, \alpha_m$ s.t.

$C_0 = \{x \in V \mid (\alpha_i, x) \geq 0, \ i = 1, \dots, m\}$

i.e. find the α's s.t. $H_{\alpha_1}, \dots, H_{\alpha_m}$ bound C_0 (and α_i have pos. projection onto points in C_0.)



<u>Def</u>: These α_i are the <u>simple roots</u> and $s_i := s_{\alpha_i}$ are the <u>simple reflections</u>.

<u>Ex.</u> from above simple roots $\vec{e}_i - \vec{e}_{i+1}$

simple reflections are <u>adjacent transpositions</u>

<u>Thm</u>: 1.) W is generated by the reflections s_i

2.) Any root $\alpha \in \Phi$ has the form $s_{i_1} s_{i_2} \cdots s_{i_{k-1}} (\alpha_{i_k})$

simple reflections — simple root

3.) Any reflection s_α has the form $s_\alpha = s_{i_1} s_{i_2} \cdots s_{i_{k-1}} s_{i_k} s_{i_{k-1}} \cdots s_{i_2} s_{i_1}$

$s_\alpha : \lambda \to \lambda - (\alpha^\vee, \lambda)\alpha$ where $\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}$

enough to know this constant for simple roots.

<u>Def</u>: <u>Cartan Matrix</u> $A = (a_{ij})$ where $a_{ij} = (\alpha_i^\vee, \alpha_j)$

indexed over simple roots.

Properties of Cartan matrix:

0. (If Crystallographic) $a_{ij} \in \mathbb{Z}$

1. $a_{ii} = 2$

2. $a_{ij} \leq 0$ if $i \neq j$

3. If $a_{ij} < 0$ then $a_{ji} < 0$

4. A is an $r \times r$ matrix where r is rank of the root system