LECTURE 8 Mon 9/22

Last Time:

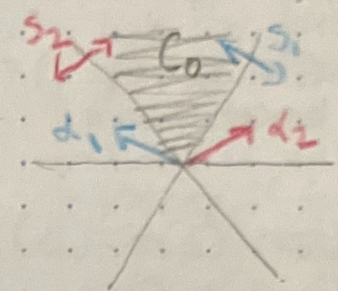
Φ root system

W Weyl (reflection) group

$\alpha_1, \ldots, \alpha_m$ simple roots will see later that $m =$ rank of root system

$s_i = s_{\alpha_i}$ simple reflections ($s_i^2 = id$)

$C_0 = \{x \mid (\alpha_i, x) \geq \forall i\}$ fundamental Weyl chamber



Walls of $C_0 = H_{\alpha_i}$

simple refl. w.r.t. walls of $C_0 = s_i$

(Once you pick C_0, simple roots & reflections determined by its walls)

<u>Thrm</u>: 1. W is generated by the s_i.

2. Any Weyl chamber has the form

$$C = s_{i_1} s_{i_2} \cdots s_{i_k} (C_0)$$

3. Any root $\alpha \in \Phi$ has the form $\alpha = s_{i_1} s_{i_2} \cdots s_{i_{k-1}} (\alpha_{i_k})$

4. Any reflection $s_\alpha \in W$ has form

$$s_\alpha = s_{i_1} s_{i_2} \cdots s_{i_{k-1}} s_{i_k} s_{i_{k-1}} \cdots s_{i_2} s_{i_1}$$ (palandromic prod. of simple refl.)

<u>Proof</u>: 2.) Induction on distance from C_0 to C

minimal # walls to cross to get from C_0 to C

<u>Base</u>: $C = C_0$.

<u>Step</u>: Assume claim true for C.

Let C' by any neighbor to C.

Weyl chamber $C' = s_\alpha (C)$

Know: $C = w(C_0)$ where w a product of simple reflections

$C' = s_\alpha w (C_0)$

may not be a product of simple chambers yet.

BUT we know we got that wall by applying reflections to one of original walls of C_0.

$\Rightarrow \alpha = w(\alpha_j)$. ← use to show (3)

$s_\alpha = w s_j w^{-1}$ ← use to show (4) (exercise: check this w/ matrix linear alg.)

$C' = s_\alpha w(C_0) = w s_j w^{-1} w(C_0)$.

↖ product of simple refl.

Then $(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$

$\boxtimes$

Recall $s_\alpha : \lambda \mapsto \lambda - (\alpha^\vee, \lambda)\lambda$ where $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ (*)

$s_i : \lambda \mapsto \lambda - (\alpha_i^\vee, \lambda)\lambda$.

<u>Cartan Matrix</u>: $A = (a_{ij})$ where $a_{ij} = (\alpha_i^\vee, \alpha_j)$ ($m \times m$ matrix)

<u>Exercise on Problem set</u>: Show m = rank of root system (i.e. dim of space it spans)

<u>Q</u>: How do we go from setting of roots & reflections to chip firing game?

<u>A-Kostant Game</u> or <u>Cartan-Kostant Game</u>

"chip config." $(c_1, c_2, \dots, c_m) \rightsquigarrow$ linear combo. of simple roots $c_1\alpha_1 + \dots + c_m\alpha_m$

Start w/ config. $(0, \dots, 0, \overset{\text{spot } j}{1}, 0, \dots, 0)$.

Step: Apply s_i's.

$\alpha = c_1\alpha_1 + \dots + c_m\alpha_m \overset{s_i}{\rightsquigarrow} c_1 s_i(\alpha_1) + c_2 s_i(\alpha_2) + \dots + c_m s_i(\alpha_m)$.

(*) $\Rightarrow s_i(\alpha_j) = \alpha_j - (\alpha_i^\vee, \alpha_j)\alpha_i = \alpha_j - a_{ij}\alpha_i$ ← entry in Cartan matrix

Then $s_i(\alpha) = c_1\alpha_1 + \dots + c_m\alpha_m - \left(\sum_j a_{ij} c_j\right)\alpha_i$

so in terms of chips...

$(c_1, \dots, c_n) \overset{s_i}{\longmapsto} (c_1, c_2, \dots, c_{i-1}, c_i', c_{i+1}, \dots, c_m)$

where $c_i' = c_i - \sum_{j=1}^{n} a_{ij} c_j = \sum_{j \neq i} (-a_{ij}) c_j - c_i$. (Recall $a_{ii} = 2$)

This is exactly Kostant's game for a directed graph where edge $i \to j$ has multiplicity $-a_{ij}$.

__Recipe:__ For each config. $(0, \ldots, 0, 1, 0, \ldots, 0)$ play the game

game stops ($\forall$ init. conditions)
$\Downarrow$
Get a root system!

Does not stop
$\Downarrow$
NOT a root system.

Note: For directed graphs, the result might depend on choice of init config. If fails to stop for __any__ init. config, it's not a root system

__Properties of A__ (The Cartan Matrix)

__Thm:__ 0. $a_{ij} \in \mathbb{Z}$ (for crystallographic root system)

1. $a_{ii} = 2$
2. $a_{ij} \leq 0$ if $i \neq j$
3. If $a_{ij} < 0$ then $a_{ji} < 0$
4. $m = r$ Equivalently: $\alpha_1, \ldots, \alpha_m$ is a linear basis ($\Leftrightarrow \det A \neq 0$).

simple roots

dim of space spanned by Φ
$= \operatorname{rank} \Phi$

These are necessary but not sufficient conditions to be a Cartan matrix.

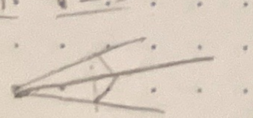
__Exercise:__ Prove 2

i.e. Show angle between any 2 simple roots is $\geq \pi/2$

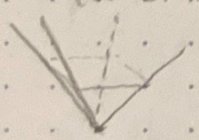
$\hookrightarrow$ Can see why it's true by restricting to plane generated by α_i & α_j

__Proof of 4:__ (4) $\Leftrightarrow$ Any Weyl chamber is a simplicial cone (a d-dim'l cone with d walls)

__Ex.__ __r=3__



simplicial cone ✓



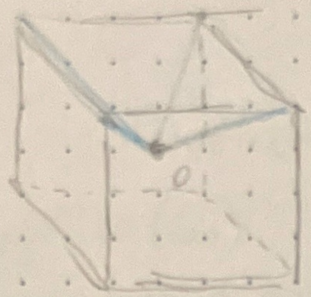
cone generated by square pyramid ✗

Counter Ex. (What goes wrong w/ non-simplicial cone)

Ex.

3-dim cube

$[-1,1]^3$



Make square pyramid from origin to each face

$\rightsquigarrow$ This isn't a hyperplane arrangement at all, just a <u>fan</u>

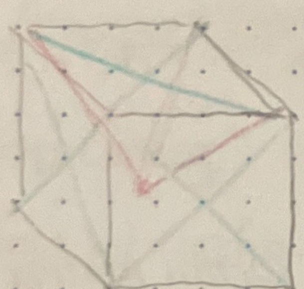
a collection of cones that cover the whole space

<u>Ex.</u>

A fan coming from a hyperplane arrangement

A fan NOT coming from hyperplane arrangement

In cube example above, you have to extend all hyperplanes. Then you get



chambers

$6 \cdot 4 = 4!$

$\Rightarrow A_3$

Another way to do A_3 root system