18.217 - Combinatorial Theory // 9.24.25

<u>(Recall)</u> Given a matrix $A = (a_{ij})$ (Cartan Matrix) of a crystallographic root system ϕ, we have that each

$$a_{ij} = (\alpha_i^\vee, \alpha_j)$$

where $\alpha_1, \dots, \alpha_r$ are simple roots, and $\alpha_i^\vee = (2/(\alpha_i, \alpha_i)) \cdot \alpha_i$

<u>(Recall) Thm:</u> for $a_{ij} \in \mathbb{Z}$ (always the case in cryst. root system)

0) $a_{ii} = 2$

1) $a_{ij} \leq 0$, for $i \neq j$

2) $a_{ij} \neq 0 \Rightarrow a_{ji} \neq 0$

3) $\det(A) \neq 0$. (exercise to prove)

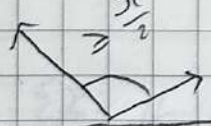
Equivalently, we can say

→ $\{\alpha_1, \dots, \alpha_r\}$ forms a linear basis

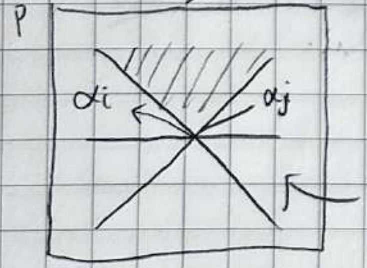
→ # of simple roots = rank of ϕ

→ Weyl chambers are simplicial cones.

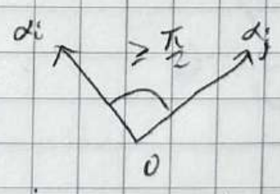
<u>(Proof) - (2):</u> Let P be the plane spanned by α_i and α_j, and intersect the Coxeter arrangement with P.



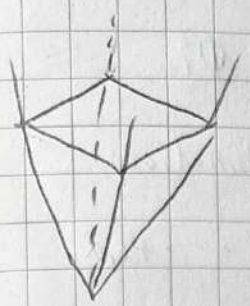
In the two-dimensional case, the chambers form an acute angle



$\Rightarrow$



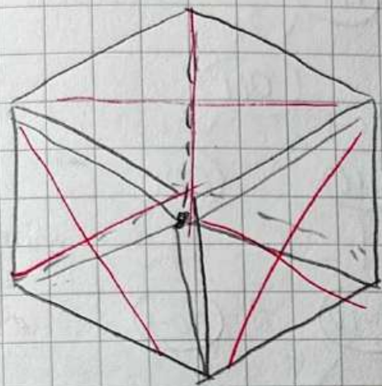
Recall that the following cannot be a Weyl chamber



Prop: the Coxeter arrangement of type A_3 can be constructed as the min. hyperplane arrangement, whose fan refines the face fan of a cube.

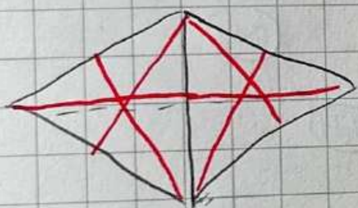
(equiv: of a tetrahedron)

EX:



hyperplanes = # edges / 2 = $\frac{12}{2} = 6$

chambers $= 6 \cdot 4 = 24$



hyperplanes = 6

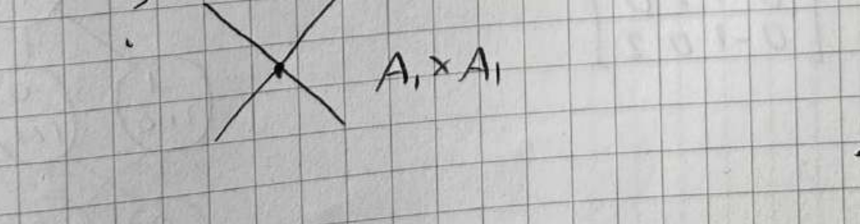
chambers = 24

*Note: a one-dimensional cube = a one dimensional tetrahedron, they are both just a line.

EX: $r=1$

type A_1

$r=2$



→

(Recall) if you play Kostant's game, for a matix $A = (a_{ij})$, we find that chip configurations for A-Kostant game correspond to $\alpha \in \Phi$ roots $\alpha \in \Phi$

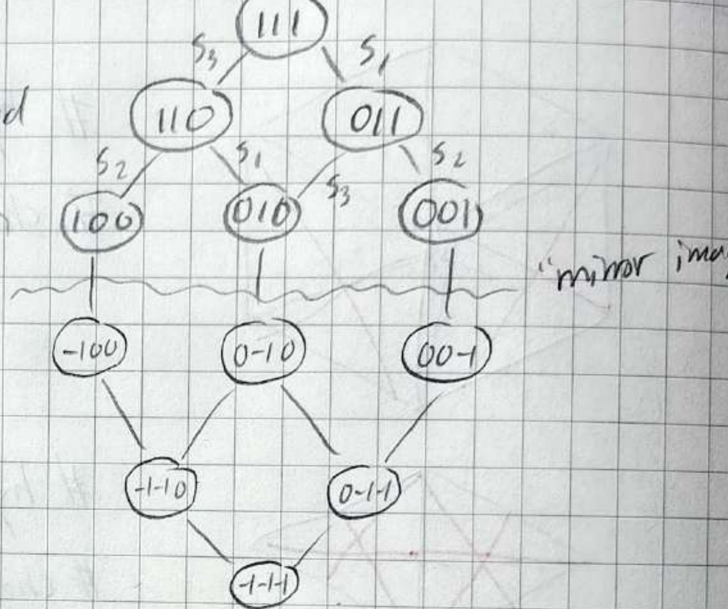
$$\{A\text{-Kostant configs}\} \leftarrow \{\text{roots } \alpha \in \Phi\}$$

we can express these chip cofigurations as linear combinations

$$(c_1, \dots, c_r) \longrightarrow \alpha = c_1\alpha_1 + c_2\alpha_2 + \dots + c_r\alpha_r$$

EX: Consider A_3, we get

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$ and



(recall the diamond and hexagon properties)

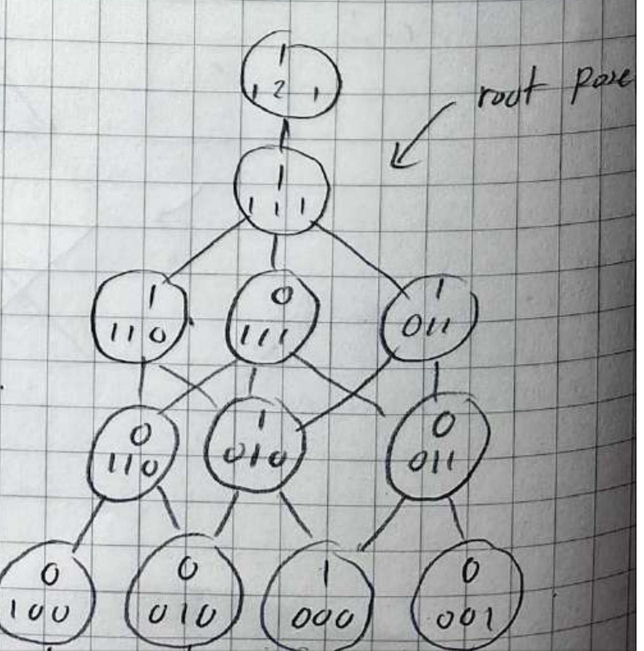
Thm: for any $\alpha = c_1\alpha_1 + c_2\alpha_2 + \dots + c_r\alpha_r$, we have either $c_1, c_2, \dots, c_r \geq 0$ or $c_1, c_2, \dots, c_r \leq 0$

Def'n: (Root Poset)

↳ the poset P on the set of positive roots, such that $\alpha \leq \beta$ iff $\beta - \alpha$ is a nonnegative linear combination of α_i's.

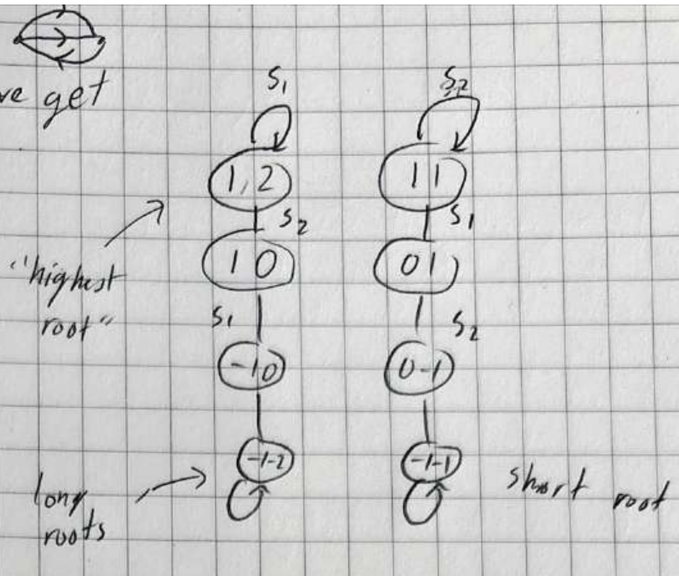
EX: consider D_4, we get

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}$$



EX: consider B_2 $\bullet \Rightarrow \bullet$, we get

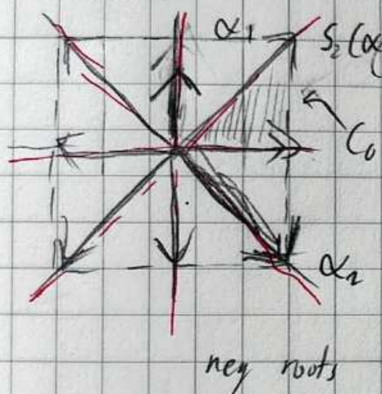
$$A = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$$



In the non-simply-laced case, we'll have exactly two connected components in the root poset if the corresponding Dynkin diagram is connected.

↳ two possible "highest roots."

Now, recall that we can construct this game w/ a square... consider B_2



* in this example, we have roots of differing lengths.

* PSET question to come on these topics.