LECTURE 10 : Fri 9/26

Recall: G simple connected undirected graph on vertex set $[n]$

Kostants game

3-types

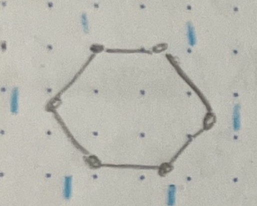
Finite Type	Affine Type	Wild Type
A-D-E	$\hat{A}$-$\hat{D}$-$\hat{E}$	everything else

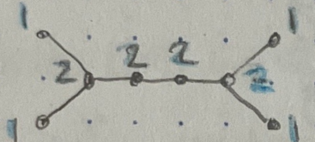
Related Construction:

Def (Vinberg): $\vec{c} = (c_1, \ldots, c_n)$ s.t. $c_i \geq 0$. Vector is

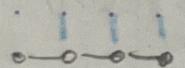
- additive if $c_i = \frac{1}{2}\sum_{j \sim i} c_j \ \forall i$ (all vertices happy, none excited)
- subadditive if $c_i \geq \frac{1}{2}\sum_{j \sim i} c_j \ \forall i$ (all vertices happy)
- strictly subadditive if $c_i > \frac{1}{2}\sum_{j \sim i} c_j \ \forall i$ (all vertices excited)

E.x.

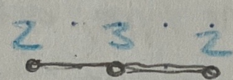




additive



← subadditive (but not strictly)



← strictly subadditive

Thrm (Vinberg): Exactly one of the following is true:

(finite) $\exists$ a strictly subadditive $\vec{c} > \vec{0}$

$$c_i > \frac{1}{2}\sum_{j \sim i} c_j \ \forall i$$

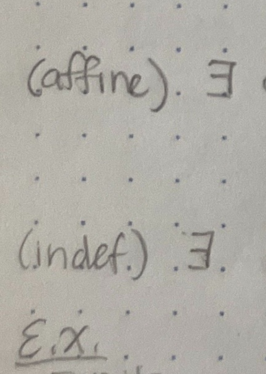
(affine) $\exists$ an additive $\vec{c} > 0$

$$c_i = \frac{1}{2}\sum_{j \sim i} c_j \ \forall i$$

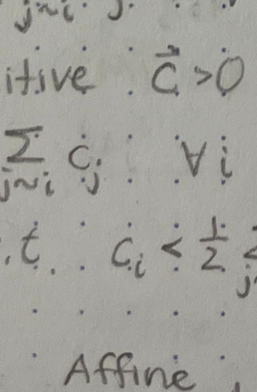
(indef) $\exists$ $\vec{c} > 0$ s.t. $c_i < \frac{1}{2}\sum_{j \sim i} c_j \ \forall i$.

E.x.

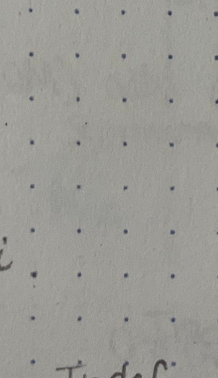
Finite



Affine



Indef



$A = (a_{ij})$ a real $n \times n$ matrix s.t.

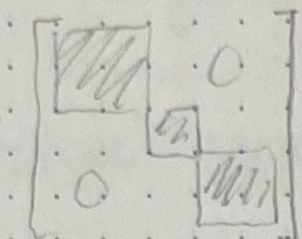
(M1) $a_{ij} \leq 0 \quad \forall\, i \neq j$

(M2) if $a_{ij} < 0$ then $a_{ji} < 0$

Cartan matrices must satisfy these conditions & more.

Def: A is indecomposable if the graph on vert. set $[n]$ with edges (i,j) for $a_{ij} < 0$ is connected.

Any A has the form

$A =$ 

indecomposable blocks

Vinbergs Thrm: A an indecomposable $n \times n$ matrix satisfying (M1) & (M2). Exactly one of the following is true

(1)
(finite): $\exists\ \vec{c} = (c_1, \ldots, c_n)^T > \vec{0}$ s.t. $A\vec{c} > \vec{0}$

(affine): $\exists\ \vec{c} = (c_1, \ldots, c_n)^T > \vec{0}$ s.t. $A\vec{c} = \vec{0}$

(Indef): $\exists\ \vec{c} = (c_1, \ldots, c_n)^T > \vec{0}$ s.t. $A\vec{c} < \vec{0}$

(2) Moreover, A^T has the same type as A

(3) Moreover, in the affine type, additive conf. $\vec{c}$ is unique up to rescaling

(4) Moreover, finite $\Leftrightarrow$ A is "positive definite"

def: All principle minors (same row & column indices) are > 0

Affine $\Leftrightarrow$ A is "positive semidefinite"

def: $\det(A) = 0$ & all other principle minors > 0

2 applications:

I.) Kac Moody algebras

A satisfies (M1), (M2) also (M0) $a_{ij} \in \mathbb{Z}$ and (M3) $a_{ii} = 2$

generalized Cartan matrix

II.) Matrix Chip firing game

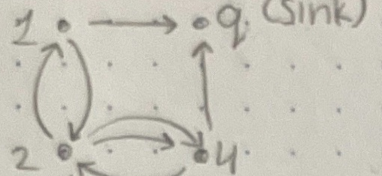
chip config. $(c_1, \ldots, c_n) \geq 0$

Moves: Fire index i: If $c_i \geq a_{ii}$ then $\vec{c} \rightsquigarrow (\vec{c} - i^{th}$ row of $A)$

(M1) gives the diamond lemma

finite: game ends. affine: goes forever but in a cycle old configs repeat. Indef: Goes forever, get bigger & bigger

Ex.



Laplacian Matrix

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & 3 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$(L)_{ii}$ = outdegree of vertex i

$(L)_{ij}$ = $-$(# edges from i to j) for $i \neq j$

$A = \tilde{L}$ = reduced laplacian

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 0 & -1 & -2 \end{bmatrix}$$ (Cross out row & column corresponding to sink vertex)

?

Assuming any time there is edge from i to j, there is an edge from j to i.

Proof idea of Vinbergs Thrm.:

Key Lemma: A satisfies (M1) & (M2) & indecomposable

$\forall \vec{c} \geq \vec{0}$, $A\vec{c} \geq \vec{0}$, we have either $\vec{c} = \vec{0}$ for $c_i > 0 \ \forall i$

Exercise: Prove this. (follows from connectivity of graph.)

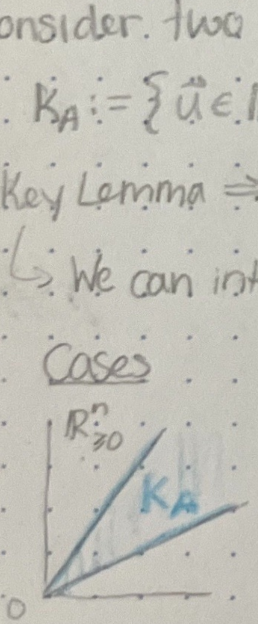
Consider two cones is $\mathbb{R}^n$

$K_A := \{\vec{u} \in \mathbb{R}^n \mid A\vec{u} \geq 0\}$

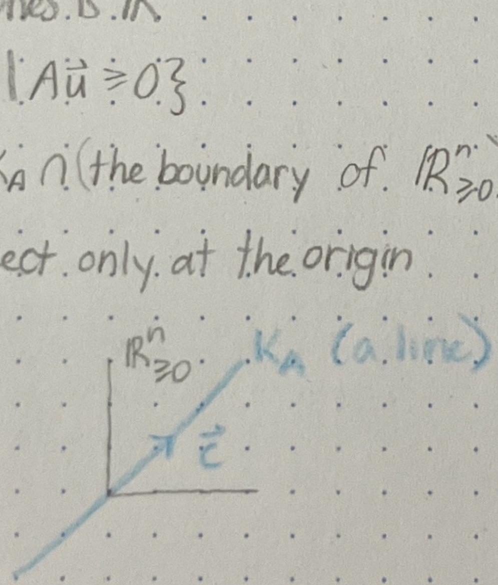
Key Lemma $\Rightarrow$ $K_A \cap$ (the boundary of $\mathbb{R}^n_{\geq 0}$) is $\vec{0}$

$\hookrightarrow$ We can intersect only at the origin

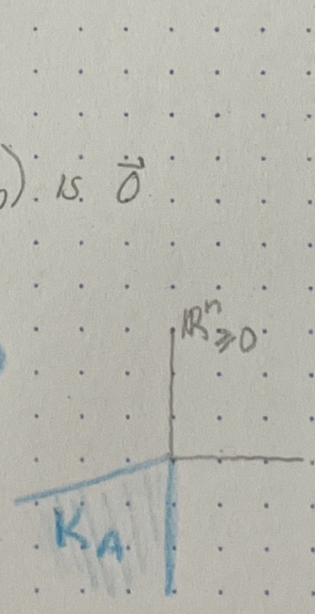
Cases



Finite



Affine



Indefinite