LECTURE 11 Mon 9/29

Last Time:

Vinberg's Thrm:

$A = (a_{ij})$ real $n \times n$ matrix s.t.

- $a_{ij} \leq 0 \ \forall i \neq j$
- $a_{ij} < 0 \Rightarrow a_{ji} < 0$
- A is indecomposable ← i.e the graph on $[n]$ w/ edges (i,j) for $a_{ij}<0$ is connected

Then exactly one of the following is true

(Finite) $\exists \ \vec{h} = (h_1, \dots, h_n) > 0$ s.t. $A\vec{h} > 0$ ← meaning >0 in every component

(Affine) $\exists \ \vec{h} > 0$ s.t. $A\vec{h} = \vec{0}$

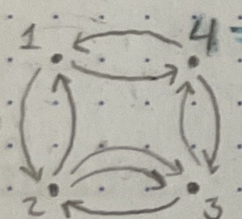
(Indefinite) $\exists \ \vec{h} > 0$ s.t. $A\vec{h} < 0$

Moreover
- A^T is of the same type as A
- If A is an integer matrix we can assume $\vec{h}$ is a integer vector.

Exercise: Prove this Thrm

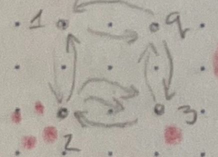
Graphical chip-firing (G-chip firing)

G a connected, directed graph s.t. if G has an edge from i to j then it also has an edge from j to i (but the multiplicities could be different)

E.x. $G =$ 

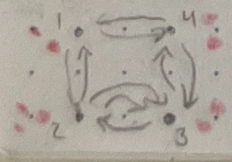
E.x. chip firing with a sink | If # chips at v exceeds out degree of v, can fire it to send chip from v to every nhbor

$\vec{c} = (c_1, c_2, c_3) \in \mathbb{Z}^3_{\geq 0}$



$(0,4,1) \xrightarrow{\text{fire } 2} (1,1,3) \xrightarrow{\text{fire } 3} (1,2,1)$ final config.

E.x. chip firing w/out a sink $\vec{c} \in \mathbb{Z}^4_{\geq 0}$



$\vec{c}_{init} = (2,3,2,2) \xrightarrow{\text{fire } 1} (0,4,2,3) \xrightarrow{\text{fire } 2} (1,1,4,3) \xrightarrow{\text{fire } 3} \xrightarrow{\text{fire } 4} (3,2,3,2)$

Oops that was a different configuration, but

Exercise: Find a different order of firing (and maybe different starting config.?) so it does go in a loop forever.

Matrix Chip Firing (A-chip firing)

A as in Vinberg's Thrm

$\vec{c} = (c_1, \dots, c_n) \in \mathbb{Z}^n_{\geq 0}$

Fire i: if $c_i \geq a_{ii}$ then

$$\vec{c} \longrightarrow \vec{c} - (i^{\text{th}} \text{ row of } A)$$

↪ no-sink firing corresponds to Laplacian matrix
↪ with sink firing corresponds to reduced Laplacian matrix.

Ex. Laplacian of graph from before:

$$L = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} \longleftarrow \text{diag.} = \text{out degree}$$

(L_{ij} for $i \neq j$) = # edges from i to j.

Reduced Laplacian

$$L' = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & 2 \\ 0 & -1 & 2 \end{pmatrix}$$

Cross out row & column corresponding to sink.

Thrm A: A-chip firing is finite $\forall$ initial configs iff A is of finite type.

Thrm G: G-chip firing with a sink is always finite.

In general, when not all vertices have edge to sink, Thrm G not obvious. Use the following:

Lemma: $\forall$ graph G as above, $\exists\, \vec{h} = (h_1, \dots, h_n) > 0$ s.t.

$$L' \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} > 0$$

(equiv. matrix $L' \cdot \operatorname{diag}(h_1, \dots, h_n)$ has strictly positive row sums)

Ex. $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1.001 & \\ & & 1 \end{pmatrix} \longleftarrow$ has all positive row sums

Lemma $\Rightarrow$ Thrm G: Use this as weighted sum of chips. Every time you fire a chip, the weighted sum decreases by at least a fixed ε (in this Ex. 0.001)

Proof of Thm A:

- If A is of finite type:

$\Rightarrow \exists\, h>0$ s.t. $Ah>0$.

Now, given a chip config. $\vec{c}=(c_1,\dots,c_n)$

let weighted chip config be $wt_h(\vec{c})=h_1c_1+\dots+h_nc_n$.

$\leadsto$ Every time you fire a chip, the weighted chip config sum goes down.

$\Rightarrow$ Eventually each vertex has too few chips to fire anymore & game ends

- If A is of affine or indet. type:

A^T is of the same type $\Rightarrow$

$\exists\, \vec{h}>0$ s.t. $A^T\vec{h}\leq 0$. $h_i\in\mathbb{Z}_{>0}\ \forall i$

$(h_1,\dots,h_n)A\leq 0 \iff h_1\,\vec{row_1}+\dots+h_n\,\vec{row_n}\leq 0$

so if all entries of $\vec{c}_{init}$ are sufficiently large,

we can fire vertex i h_i times $\forall\, i=1,\dots,n$

and get a config. $\vec{c}\geq\vec{c}_{innit}$ (termwise).

$\Rightarrow$ We can keep doing this forever, so game is not finite.

What was wrong with earlier e.x.?

We fired each vertex once.

$\vec{h}=(1,1,1,1)$ is a vector s.t. $A\vec{h}=0$, but we need to find a vector $\vec{h}$ s.t. $A^T\vec{h}=0$, and then fire each vertex the corresponding # times to that.

Observation: The type of $A\cdot\mathrm{diag}(\lambda_1,\dots,\lambda_n)$ $\lambda_i>0$ is the same as the type of A.

The type for matrix is same as graphical chip firing (modulo rescaling)