LECTURE 12 : Wed 10/1

Recall

Vinberg's Thrm $A = (a_{ij})$.

$a_{ij} \le 0$, $a_{ij} < 0 \iff a_{ji} < 0$, A indecomposable

exactly one of the following holds.

finite type: $\exists \vec{h} > \vec{0}$ s.t. $A\vec{h} > 0$

Affine type: $\exists \vec{h} > \vec{0}$ s.t. $A\vec{h} = \vec{0}$

Indefinite Type: $\exists \vec{h} > 0$ s.t. $A\vec{h} < \vec{0}$

Special Case: Generalized Cartan matrices:
Additionally require all $a_{ij} \in \mathbb{Z}$ and $a_{ii} = 2$

finite: A, B, C, D, E, F, G

affine: $\tilde{A}, \tilde{B}, \ldots, \tilde{G}$

Problem: Classify finite/affine types when $a_{ii} \le 3$ $\forall i$

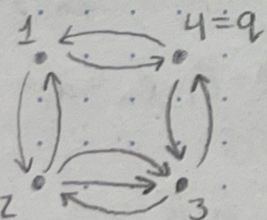
Another Special Case: Graphical case: Additionally require all row sums ≥ 0

At Least one row sum $\gneq 0$

$\Rightarrow$ finite type

- $A = L'$ reduced Laplacian matrix of some graph G
- G chip firing w/ a sink

Ex.

$G =$ 

$$L = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

$$L \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \vec{0}$$

Chip firing without sink

Can play game to get a loop back to where we started.
Fire each edge according to vector $\vec{h}$ s.t. $L^T\vec{h} = 0$ $\rightsquigarrow$ $\vec{h} = \begin{pmatrix} 5 \\ 4 \\ 7 \\ 0 \end{pmatrix}$

i.e fire vertex 1 five times, vertex 2 four times & vert. 3 seven times to get back to the chip configuration you started with

Game never ends, but finite in sense that from a given initial config, only finitely many configs we can get to.

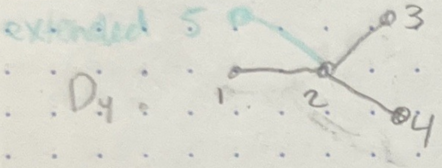
<u>Lemma</u>: $\forall$ finite (resp. affine) A, $\exists$ diag matrix $D = \mathrm{diag}(d_1, \dots, d_n)$, $d_i \geq 0$ s.t. AD is L' (resp L'), the reduced Laplacian matrix for some graph.

<u>Proof sketch</u>: The $\vec{h}$ from Vinberg's Thm tells you the rescaling.

$$\rightsquigarrow D = \mathrm{diag}\left(\frac{1}{h_1}, \frac{1}{h_2}, \dots, \frac{1}{h_n}\right)$$

<u>Ex.</u> Cartan matrix A (resp. extended Cartan matrix $\tilde{A}$) of type D_4

D_4:



$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \qquad \tilde{A} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & -1 \\ 0 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

Note not all row sums are ≥ 0.

Find additive function on

(everyone is happy, no one is excited)

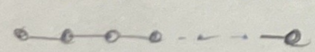
$$\Longrightarrow D = \begin{pmatrix} 1 & & & & 0 \\ & 2 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix} \qquad \tilde{A}D = \begin{pmatrix} 2 & -2 & 0 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 & 2 \end{pmatrix}$$

All row sums are 0 now.

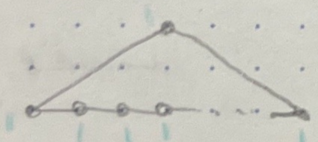
Equivalent to chip firing on this graph

<u>Hypersimplicies, Eulerian Numbers & Their generalization to Root Systems</u>

A_r:



$\tilde{A}_r$



$(r+1)$-cycle

additive function

simple roots: $\alpha_i = \vec{e}_i - \vec{e}_{i+1} \in \mathbb{R}^n/(1, \dots, 1)\mathbb{R}$ for $i = 1, \dots, r$

$r = n-1$

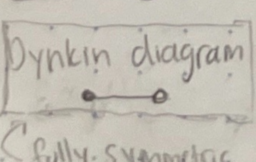
Highest root: $\theta = \vec{e}_1 - \vec{e}_n = \vec{\alpha}_1 + \dots + \alpha_r$

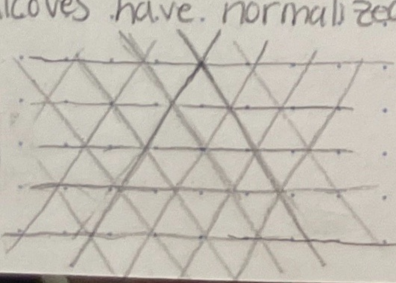
<u>Coxeter arrangement</u>: $x_i - x_j = 0$ $i \neq j \in [n]$ hyperplanes in $\mathbb{R}^n/(1, \dots, 1)\mathbb{R}$

<u>Affine coxeter arr.</u>: $x_i - x_j = k$ $i \neq j \in [n]$, $k \in \mathbb{Z}$ "

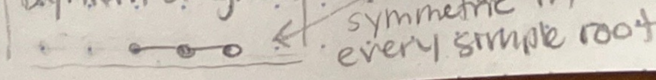
<u>FACTS</u>: These affine hyperplanes subdivide the space into simplices called the <u>alcoves</u>. All alcoves have normalized volume 1

<u>Ex.</u> A_2





A_3: subdivide 3D space with tetrahedra. (no longer regular tetrahedron)



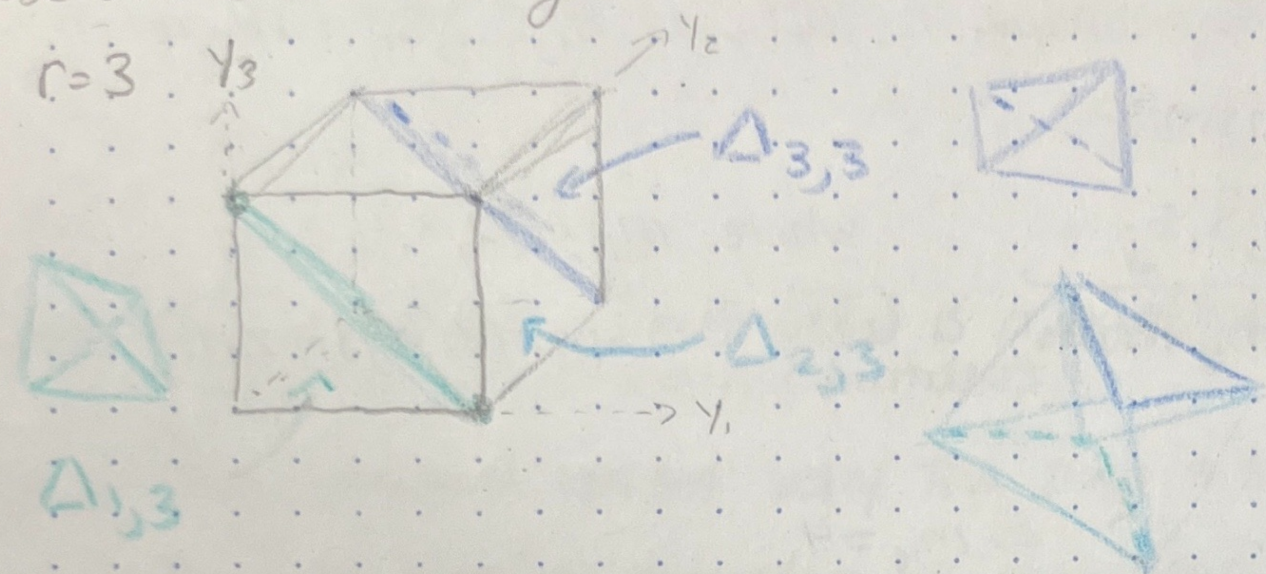
k^{th} hypersimplex

$$\Delta_{r,k} := \left\{ \vec{x} \;\middle|\; \begin{matrix} 0 \le (\alpha_i, \vec{x}) \le 1 \\ k-1 \le (\theta, \vec{x}) \le k \end{matrix} \right\} = \left\{ (y_1, \dots, y_r) \;\middle|\; \begin{matrix} 0 \le y_i \le 1 \\ k-1 \le y_1 + \dots + y_r \le k \end{matrix} \right\}$$

$$(\alpha_i, \vec{x}) \longrightarrow y_i$$

These are slices of higher dim'l cube

Ex. $r=3$



Exercise: how is ← this subdivided into alcoves

Eulerian # = normalized volume of $\Delta_{k,r}$.

$\Delta_{k,r}$ consists of the Eulerian # of alcoves.