LECTURE 13   Fri 10/3

Φ any root system (not nec. crystollographic)

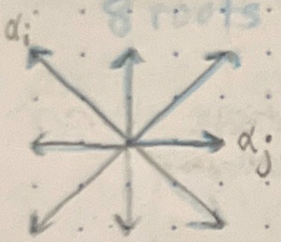
$\alpha_1, \dots, \alpha_r$ simple roots, $s_1, \dots, s_r$ simple reflections.

<u>W weyl group</u> = group generated by reflections s_α, $\alpha \in \Phi$

<u>Thm</u>: W is generated by simple reflections $s_1, \dots, s_r$ with relations

(Coxeter Relations)
- $s_i^2 = 1 \quad \forall i$
- $\underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ terms}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ terms}}$ where $m_{ij} \in \{2, 3, 4, \dots\}$ s.t. $\Phi \cap \operatorname{span}(\alpha_i, \alpha_j) = 2m_{ij}$

<u>Ex.</u>



$\leftarrow 2m_{ij} = 8 \Rightarrow m_{ij} = 4$

2-dim rt. system in Φ

In Crystallographic Cases

$$m_{ij} = \begin{cases} 2 \text{ if } & A_1 \times A_1 \\ 3 \text{ if } & A_2 \\ 4 \text{ if } & B_2 \\ 6 \text{ if } & G_2 \end{cases}$$

<u>Ex.</u> A_r: $W = S_n$

$s_i^2 = 1$, $s_i s_j = s_j s_i$ $|i-j| \geq 2$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

B_r W = hyperoctahedral group
= the group of symmetries of r-cube
= the group of signed permutations

s_i = reflect w.r.t. $x_i - x_{i+1} = 0$ for $i = 1, \dots, r-1$
s_r = reflect w.r.t. $x_r = 0$

same rels as A_r except with $s_{r-1} s_r s_{r-1} s_r = s_{r-1} s_r s_{r-1} s_r$

C_0 fundamental Weyl chamber, C any Weyl chamber.

Prop: Any two products of simple reflections s.t.

$$C = s_{i_1} s_{i_2} \cdots s_{i_k}(C_0) = s_{j_1} s_{j_2} \cdots s_{j_k}(C_0)$$

can be obtained from each other by the moves

(1) $\underbrace{\quad}_{A}\, s_i s_i \underbrace{\quad}_{B} \longrightarrow \underbrace{\quad}_{A}\,\underbrace{\quad}_{B}$ (apply $s_i^2 = Id$)

(2) $\underbrace{\quad}_{A}\, \underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ terms}} \underbrace{\quad}_{B} \longrightarrow \underbrace{\quad}_{A}\, \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ terms}} \underbrace{\quad}_{B}$

Corollary: The map $\underset{\in}{\overset{W}{w}} \longmapsto w(C_0)$ is a bijection between W & the set of Weyl chambers.

Proof: We showed earlier why the map is injective (can get any chamber)

Now we know it's surjective b/c prop. tells us if we have different $s_{i_1} \cdots s_{i_k}$ & $s_{j_1} \cdots s_{j_k}$ sending C_0 to the same chamber,

these are equivalent via Coxeter group relations (so same w)

Prop $\Rightarrow s_{i_1} \cdots s_{i_k} = s_{j_1} \cdots s_{j_k}$

Prop $\Rightarrow$ Thrm

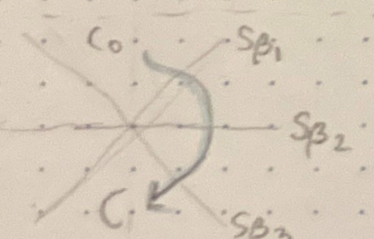
Proof Sketch of Prop. $w = s_{i_1} s_{i_2} \cdots s_{i_k} =$

$$\cdots \underbrace{(s_{i_1} s_{i_2} s_{i_3} s_{i_2}^{-1} s_{i_1}^{-1})}_{= s_{\beta_3}} \underbrace{(s_{i_1} s_{i_2} s_{i_1}^{-1})}_{= s_{\beta_2}} \underbrace{s_{i_1}}_{= s_{\beta_1}}$$

$$\begin{cases} \beta_1 = \alpha_{i_1} \\ \beta_2 = s_{i_1}(\alpha_{i_2}) \\ \beta_3 = s_{i_1} s_{i_2}(\alpha_{i_3}) \\ \text{etc} \end{cases}$$

Slight aside: Hurwitz action on decompositions in any group

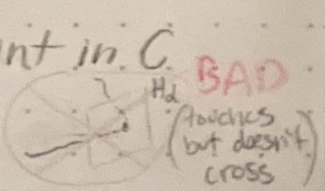
$\cdots a \cdot b \cdots = \cdots (a b a^{-1}) a \cdots$



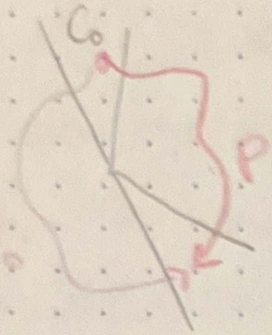
Any decomp. $w = s_{i_1} \cdots s_{i_k}$ corresponds to a walk on alcoves $C_0 \to C_1 \to C_2 \to \cdots \to C$ s.t. C_{i-1} & C_j are adjacent alcoves separated by the hyperplane H_{β_i}

Pick any "generic" continuous path from a pt. in C_0 to a point in C

↳ (1) path is transversal (i.e. not tangent) to all H_α

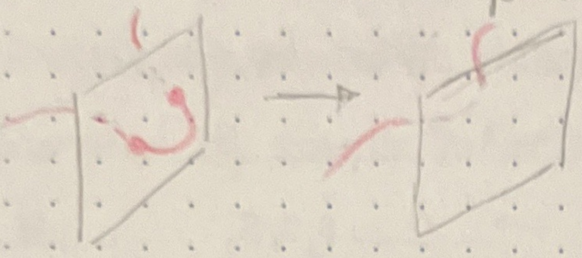


(2) Path does not pass through codim 2 intersection of H_α's (can't cross intersection of 2 or more hyperplanes)



Want to transform one path into the other.

Local moves on paths:

(1) 

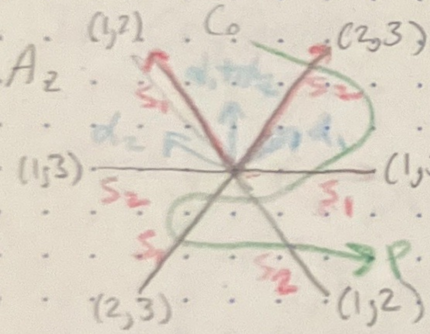
$s_i^2 = id$

(2) S 

$\underbrace{s_i s_j \cdots}_{m_{ij}} = \underbrace{s_j s_i \cdots}_{m_{ij}}$

Codim S = 2
(intersection of high dim. arr w/ a plane.)

Why do we care about codim. 2 only?
When we deform a line, we get a 2 dim'l surface which sometimes crosses codim. 2 spaces, but never crosses lower dim. space in the generic case.

A_2 

A different labelling of wall

simple transpositions

$p = s_2 s_1 s_2 \cancel{s_1 s_1} s_2$

$= s_{12} s_{23} s_{12} s_{13} s_{23}$ ⟵ non-simple transpositions

$\alpha_1 = e_1 - e_2 \rightsquigarrow$ label s_{α_1} by (1,2)

$\alpha_2 = e_2 - e_3 \rightsquigarrow$ label s_{α_2} by (2,3)

$\alpha_3 = e_1 - e_3 \rightsquigarrow$ label s_{α_3} by (1,3)