18.217 - Combinatorial Theory / 10/6/25

(Recall) say Φ is a root system, and $s_1, \ldots, s_r$ are simple reflections

→ ~~A~~ Weyl group W is generated by the simple reflections (s_i) and Coxeter relations, with two labellings of walls between chambers in the Coxeter arrangement

Dually, we can describe this as the...

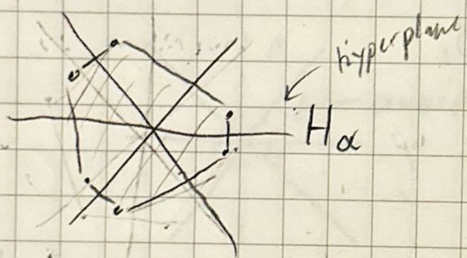
<u>W-permutohedron:</u>

↳ Fix λ in the interior of chamber C_0 ($\lambda \notin H_\alpha$, $\forall \alpha \in \Phi$). Then, we define

$$P = P_W(\lambda) = \mathrm{Conv}\left(w(\lambda) \mid w \in W\right)$$

as the W-permutohedron.

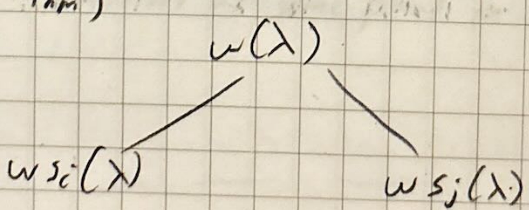
<u>EX:</u> for A_2, we have

$P =$ 

<u>Thm:</u> Vertices $w(\lambda)$ are in bijection w/ elts $w \in W$, and two vertices $u(\lambda)$ and $w(\lambda)$ are connected by an edge iff

$$u = ws_i. \leftarrow \text{simple reflection}$$

<u>EX:</u> (diagram of thm)



Last time, we discussed walks on Weyl chambers, and now we can discuss walks on the "skeleton" (edges) of the permutohedron.

Two Labellings:

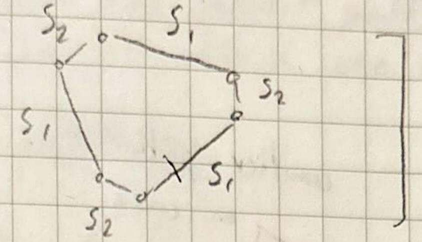
→ Right Labels: simple reflections s_i $\quad w \overset{s_i}{\text{——}} ws_i$

→ *Left Labels: arbitrary reflections $\quad w \text{ —— } s_\beta w$

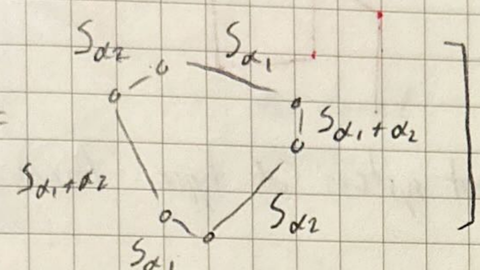
*arbitrary reflections are obtained via conjugation of simple reflections.

~~From this point of view~~

EX: permutahedron for A_2:

$P =$ 

right labels

$P =$ 

*left labels

Note:
↳ the 1-skeleton of a polytope is the set of all vertices and all edges in a polytope P.

*recall that each edge of P crosses a hyperplane H_α

From here, we find that any decomposition $w = s_{i_1} \cdots s_{i_k}$ corresponds to a walk in the 1-skeleton on of P, from vertex λ to vertex $w(\lambda)$

$$\lambda \xrightarrow[s_{\beta_1}]{s_{i_1}} s_{i_1}(\lambda) \xrightarrow[s_{\beta_2}]{s_{i_2}} s_i s_2(\lambda) \longrightarrow \cdots \xrightarrow[s_{\beta_k}]{s_{i_k}} w(\lambda)$$

where $w(\lambda) = s_{\beta_k} s_{\beta_{k-1}} \cdots s_{\beta_1}$

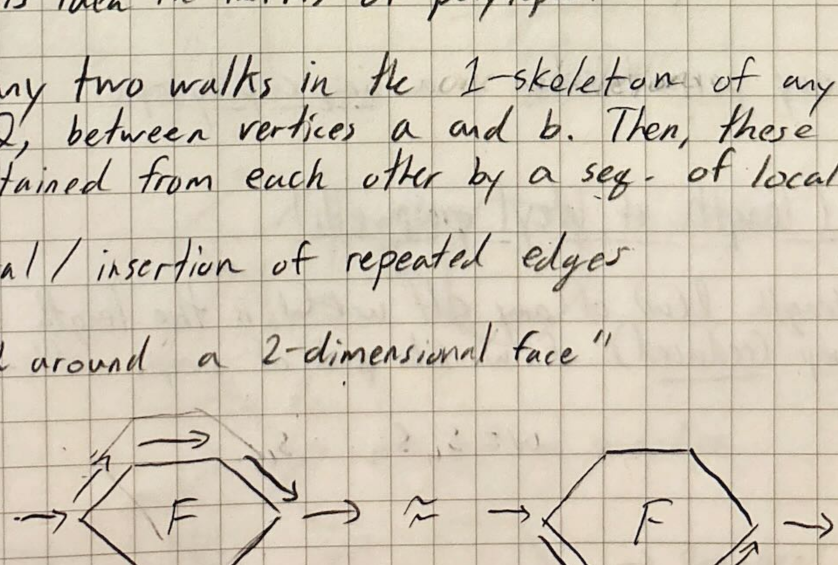
we can rephrase this idea in terms of polytopes:

Thm: look at any two walks in the 1-skeleton of any convex polytope Q, between vertices a and b. Then, these walks can be obtained from each other by a seq. of local moves:

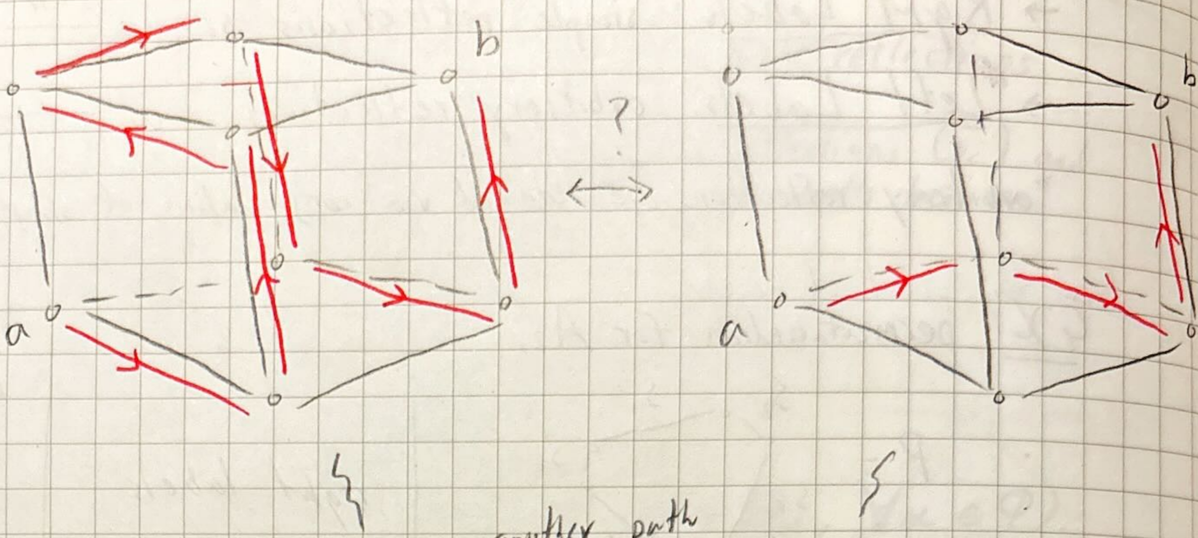
1) removal / insertion of repeated edges

2) "swith around a 2-dimensional face"

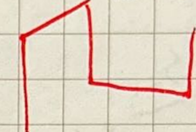
EX:



EX: consider a cube

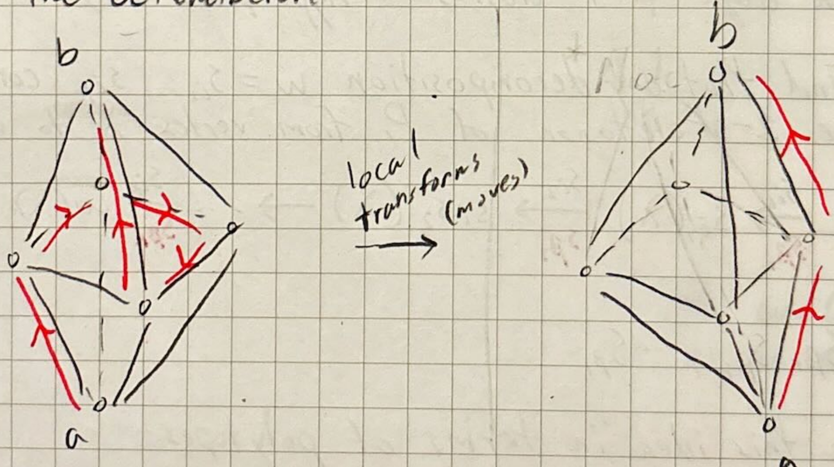


another path on cube



*Note: the cube corresponds to the root system of type $A_1 \times A_1 \times A_1$

EX: consider the octohetdron



†Note: any permutehedron is a simple polytope

Def'n (length of Weyl group elts):

↳ the length $\ell(w)$ of any elt $w \in W$ is the length of the shortest decomp (reduced) of w into prod. of simple reflections

$$w = s_{i_1} s_{i_2} \cdots s_{i_\ell}.$$

→

<u>Thm:</u> Any two <u>reduced</u> decompositions

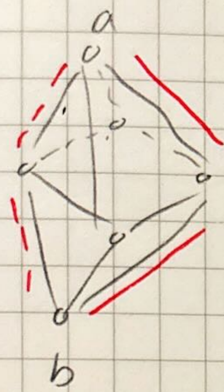
$$w = s_{i_1} \cdots s_{i_\ell} = s_{j_1} \cdots s_{j_\ell}$$

can be obtained from each other via a sequence of Coxeter moves of type (2). (w/o using the relation $s_i^2 = id$)

How can we understand this theorem in terms of polytopes?

↳ if we're given two shortest paths on a 1-skeleton from vertex a to b, can we obtain one from the other via local moves?

<u>NO</u> for octahedron



no local moves

<u>Def'n (Minkowski sum)</u>

↳ for $A, B \in \mathbb{R}^n$

$$A + B = \{a+b \mid a \in A, b \in B\}$$

Minkowski

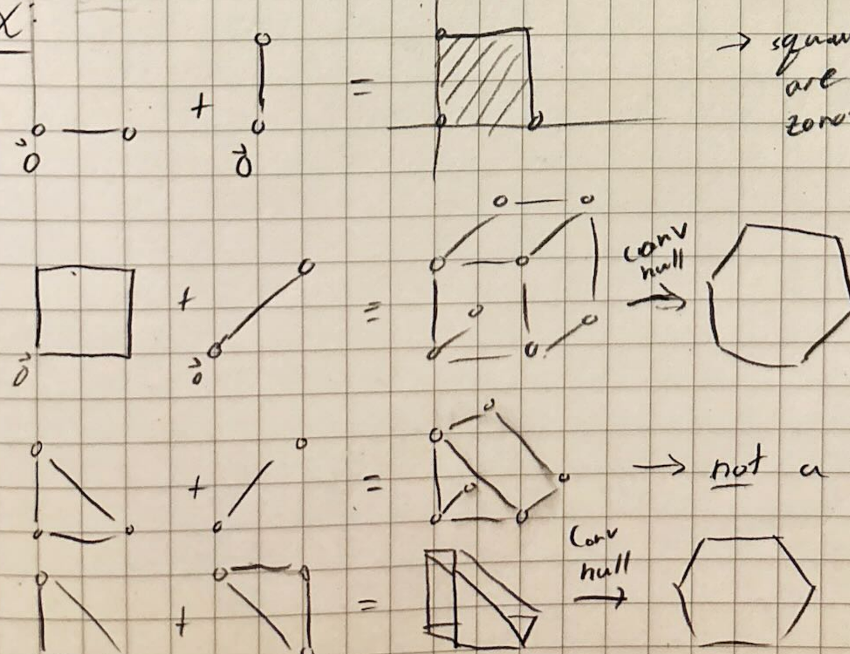
*consider how this works for >2 elements of $\mathbb{R}^n$

<u>YES</u> for *Zonotopes

<u>Def'n (Zonotopes):</u>

↳ Polytopes given by <u>Minkowski sums</u> of line segments.

<u>EX:</u>



→ squares are zonotopes!

→ zonotope!

→ <u>not</u> a zonotope

→ Zonotope!