LECTURE 25 Mon 11/3

One more pset presentation (and then normal lecture after)

<u>Problem 2</u>: (Anthony)

Show if a matrix A is finite type then A^T is.

From class: A fin type $\iff \exists \vec{h} > 0$ s.t. $A\vec{h} > 0$

Strategy: Use linear programming duality

max $c^T x$	min $b^T y$
s.t. $Ax \leq b$	$A^T y \geq c$
$x \geq 0$	$y \geq 0$

duality tells us sltns to these problems correspond

$$A = A$$
$$b = A(w - \vec{1})$$
$$c = -A^T\vec{1} + \vec{1}$$

want to end up with $y > 0$ not just ≥ 0

so add $\vec{1}$ to y

$$A^T y \geq -A^T\vec{1} + \vec{1}$$
$$A^T(y + \vec{1}) \geq \cancel{-A^T\vec{1}} + \vec{1} + \cancel{A^T\vec{1}} > \vec{0}$$

Last step: Show both sides are feasible (i.e. one side can't be unbounded)

$$A^{-1}_{ij} \geq 0$$

WTS

$$0 \leq x \leq A^{-1}b \quad (\text{this would give } x \text{ is bounded})$$

Need to have a bounded A^{-1}

$$A = s(I - B)$$

$$s > 0, \quad B_{ij} > 0$$

Apply (Frobenius Thrm??) $\Rightarrow \rho(B) = \min_x \max_i \frac{(Bx)_i}{x_i} < 1$

$$A^{-1} = \frac{1}{s}(I + B + B^2 + B^3 + \dots)$$

Another possible proof strategy:

Show finite type $\Leftrightarrow$ All principle minors are positive

same row & column indices for minor

Partitions vs. Weights

Partitions of N. $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ $\lambda_i \in \mathbb{Z}$

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0$ (Assume $\lambda_i = 0$ for $i > \ell$).

$\lambda_1 + \lambda_2 + \dots + \lambda_\ell = N$

Dominance order on partitions of N:

Partial order $\leq_N$ given by

$$\lambda \geq \mu \Leftrightarrow \lambda_1 + \dots + \lambda_i \geq \mu_1 + \dots + \mu_i \quad \forall\, i = 1, 2, \dots$$

warning: In this lecture we will use words "dominance" and "lattice" in 2 different ways

Rmk: Conjugacy classes of nilpotent elts. $\xleftrightarrow{\text{bij}}$ partitions

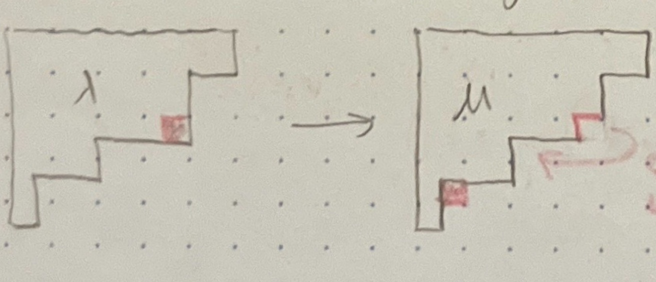
decompose to Jordan normal form & get partition from size of blocks

$$S_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$$

Schur poly, Kotska #'s, monomial symmetric functions

$K_{\lambda\mu} \neq 0$ iff $\lambda \geq \mu$

Lemma: The dominance order is generated by the following relation.

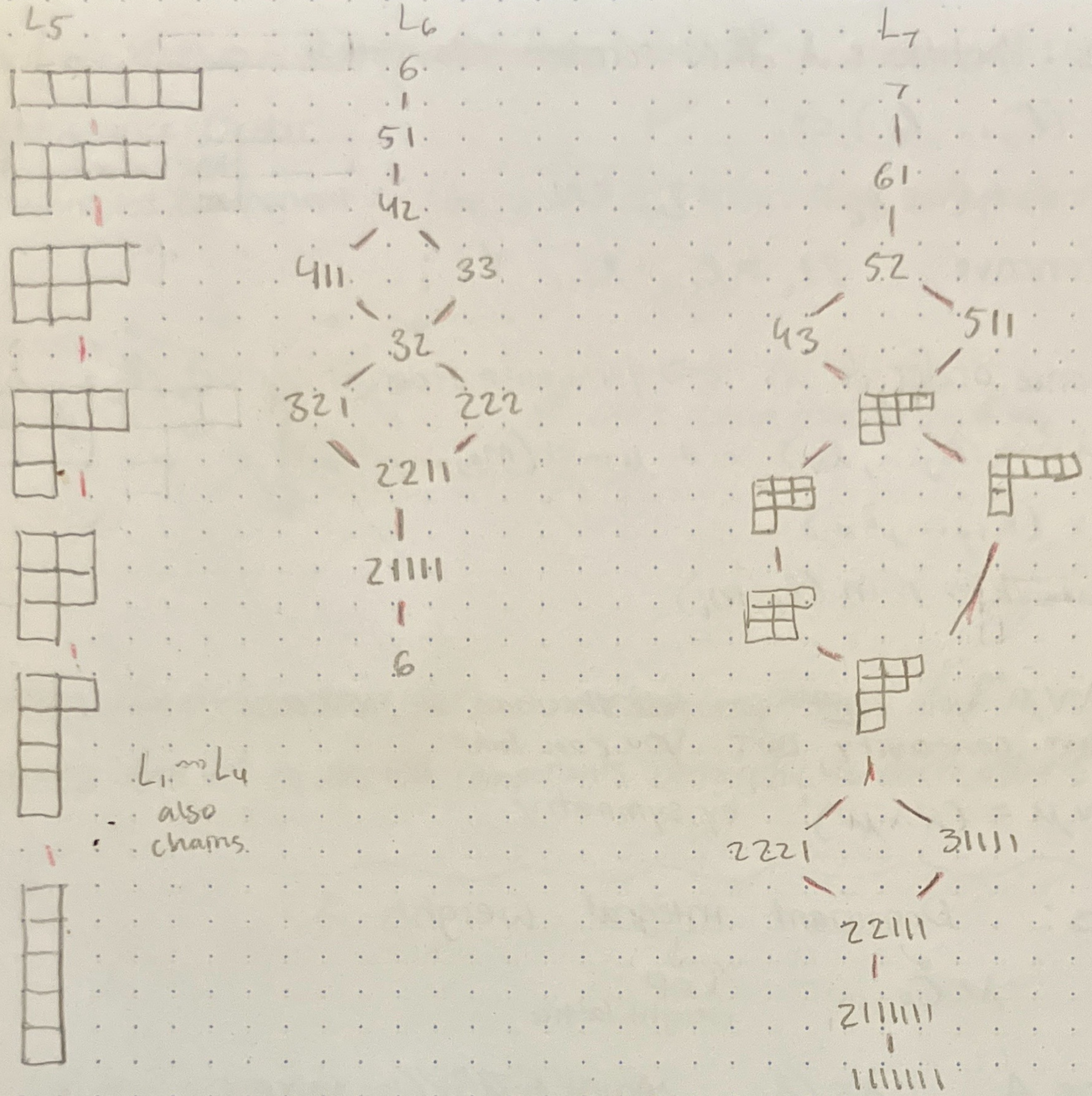


covering relation would be to put box here

And covering relations are taking a box and moving it to highest possible spot below

Cor: $\lambda \geq \mu$ iff $\lambda' \leq \mu'$

e.x.



<u>Thrm</u>: L_N is a <u>lattice</u>.

<u>Def</u>: A poset is a <u>lattice</u> that has <u>meet</u> "$\wedge$" and <u>join</u> "$\vee$" operations

$\lambda \wedge \mu =$ the unique max elt. ν s.t. $\nu \leq \lambda$ & $\nu \leq \mu$

$\lambda \vee \mu =$ the unique min. elt ν s.t. $\nu \geq \lambda$ & $\nu \geq \mu$

What are meets/joins in partitions dominance poset?

$\lambda \rightsquigarrow (\ell_1, \ell_1, \dots, \ell_N)$ where

$\ell_i = \lambda_1 + \lambda_2 + \dots + \lambda_i$

Lemma: Partitions λ of N correspond to integer vectors

$(l_0, \dots, l_N)$ s.t.

(1) $0 = l_0 \leq l_1 \leq l_2 \leq \dots \leq l_N = N$

(2) concave $\quad 2l_i \geq l_{i-1} + l_{i+1} \quad \forall i$

Dominance order on λ's $\iff$ term-wise order of vectors $(l_0, \dots, l_N)$

For $\lambda \rightsquigarrow (l_0, \dots, l_N)$ and $\mu \rightsquigarrow (m_0, \dots, m_N)$

$\lambda \wedge \mu = (k_1, \dots, k_N)$

where $k_i = \min(l_i, m_i)$

For $\lambda \vee \mu$ you can't just take the max b/c it doesn't always preserve concavity, BUT you can take

$\lambda \vee \mu = (\lambda' \wedge \mu')'$ by symmetry

Weights: Dominant integral weights λ

$\lambda \in C_0 \qquad \lambda \in P$ weight lattice

In type A_{n-1}, $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n / (1, \dots, 1)\mathbb{Z}$

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$

Similar to partitions BUT λ_i's can be negative & fix the # of them & equivalence under adding $(1, \dots, 1)\mathbb{Z}$

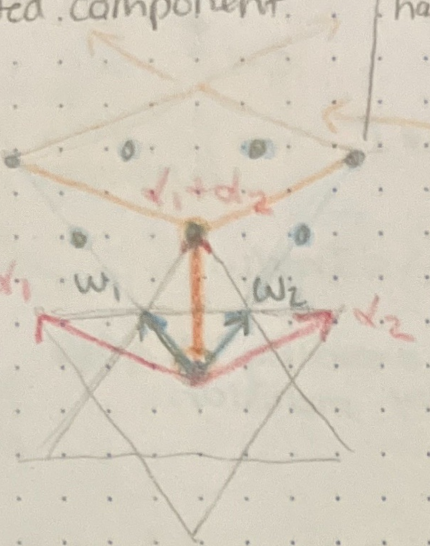
Note: If we add appropriate multiple of $(1, \dots, 1)$ to make last entry 0, then can identify them w/ partitions w/ upper bound on # of parts.

Let P^+ be the partial order on all dominant integral weights where $\lambda \geq \mu$ iff $\lambda - \mu$ is a non-neg. linear combo of positive roots

Differences between dominance order and P^+:

Dominance Order	P^+
On finite set	infinite
1 connected component	has $f := [P : Q]$ connected components

Type A_2



one connected component of P^+

P^+ cone generated by ω_1 & ω_2

<u>Problem</u>: Is it true that each connected component of P^+ is a lattice?

<u>Problem</u>: Are all connected components isomorphic to each other?

"I'm trying to prove only trivial things.
All hard things will be in the problem set."