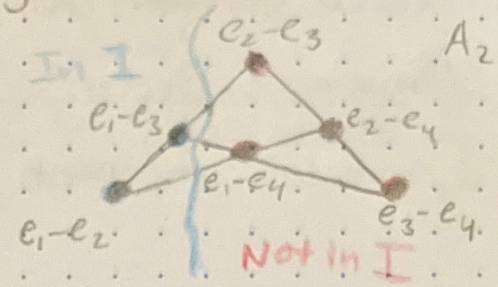
LECTURE 26 Wed 11/15



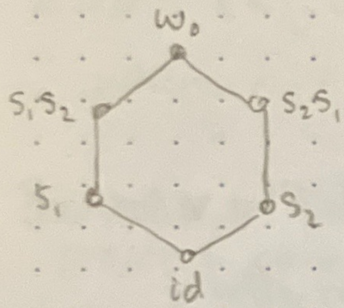
$w \mapsto Inv(w)$

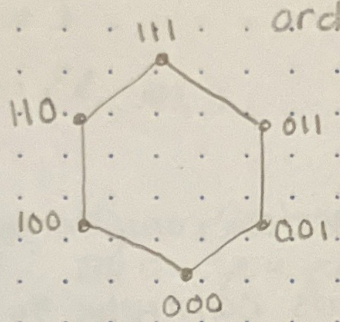
$W \xleftrightarrow{bij} \{$ subsets $I \subset \Phi^+$ that avoid root triple patterns 101 & 010 $\}$

$\exists\ \alpha, \alpha+\beta, \beta \in \Phi^+$ s.t. $\alpha, \beta \in I$ but $\alpha+\beta \notin I$

$u \leq w$ in weak Bruhat order iff $Inv(u) \subseteq Inv(w)$

Weak order ~ order on subsets of Φ^+ avoiding 101 & 010 ordered by inclusion





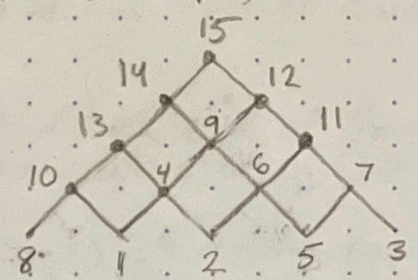
Reduced decomp of $w_0 \xleftrightarrow{coresp.}$ max'l chains from id to w_0

<u>Edelman-Green correspondence</u>

Type A_{n-1} $\left\{\begin{array}{c}\text{Reduced decomp} \\ \text{of } w_0 \in S_n\end{array}\right\} \xleftrightarrow{EG} SYT(n-1, n-2, \ldots, 1)$ (staircase shape)

How can we generalize this to other types?

$SYT(n-1, n-2, \ldots, 1) \longleftrightarrow$ linear extensions of root poset Φ^+



<u>Def</u>: For a finite poset P, an <u>order ideal</u> $I \subseteq P$ is a set s.t.

$x \in I$ & $y \leq x \Rightarrow y \in I$

$J(P)$ is the poset of order ideals ordered by inclusion.

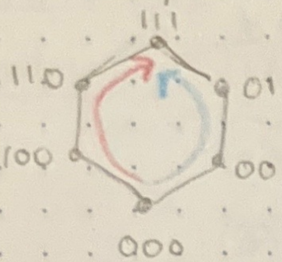
<u>Lemma</u>: Linear extensions of P are in bijection with maximal saturated chains in $J(P)$

↳ start w/ $\emptyset$ and add 1 elt at a time to get everything

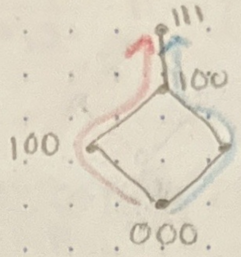
Lemma: A subset $I \subset \Phi^+$ is an order ideal in the root poset $(\Phi^+, <)$ iff it avoids root triple patterns 010 & 110.

i.e. if $\alpha+\beta \in I$ then α and $\beta \in I$.

$\left\{ \begin{array}{c} \text{Subsets of } \Phi^+ \\ \text{avoiding 010 \& 101} \\ \text{ordered by inclusion} \end{array} \right\}$



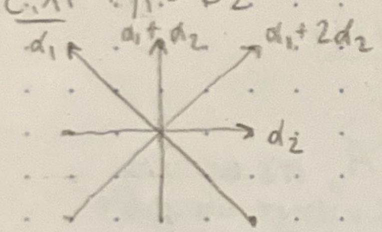
$\left\{ \begin{array}{c} \text{Subsets of } \Phi^+ \\ \text{avoiding 010 \& 110} \\ \text{ordered by inclusion} \end{array} \right\}$



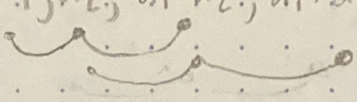
EG: The # saturated chains of these posets are the same in type A.

Problem: Are they still the same for other types?

Ex. Type B_2

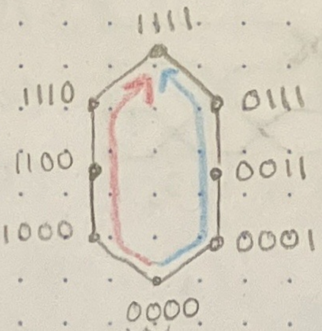


$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, \alpha_1+2\alpha_2\}$

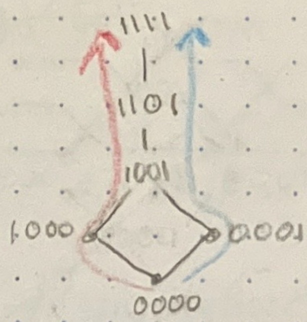


Root triples

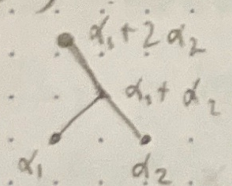
Weak Bruhat order



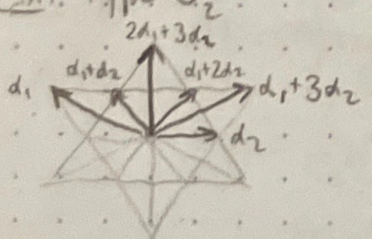
$J(\Phi^+)$



$(\Phi^+, >)$

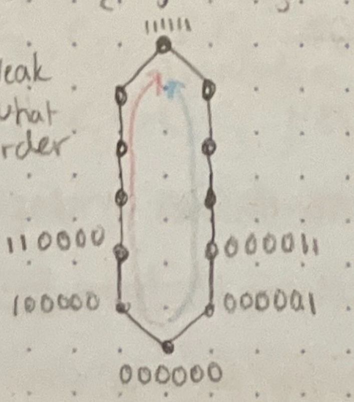


Ex. Type G_2

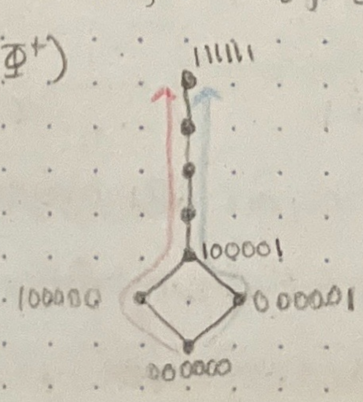


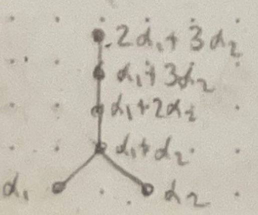
$\Phi^+ = \{\alpha_1, \alpha_1+\alpha_2, 2\alpha_1+3\alpha_2, \alpha_1+2\alpha_2, \alpha_1+3\alpha_2, \alpha_2\}$

Weak Bruhat order



$J(\Phi^+)$





__Thm__:

(1) $|W| = \prod_{i=1}^{r} (e_i + 1)$

where $e_1, \dots, e_r$ the exponents of W

e.g. $1, 2, 3, \dots, n$ in type A.

$1, 3, 5, \dots$ in type B

(2) $\# J(\Phi^+) = Cat(W) = \prod_{i=1}^{r} \frac{e_i + 1 + h}{e_i + 1}$ where h = coxeter number $= e_r + 1$.

Commutation classes of reduced decomps.

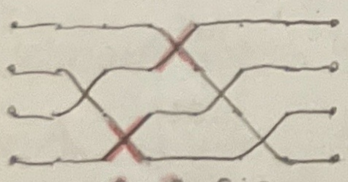
__Def__: Two red. decomp. $w = s_{i_1} \cdots s_{i_\ell}$ & $w = s_{j_1} \cdots s_{j_\ell}$, $w \in W$.

in same commutation class iff can be obtained from each other by a sequence of moves

$s_{i_1} \cdots s_{i_k} s_{i_{k+1}} \cdots s_{i_\ell} \rightsquigarrow s_{i_1} \cdots s_{i_{k+1}} s_{i_k} \cdots s_{i_\ell}$

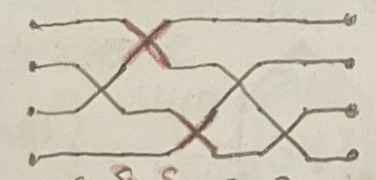
when $s_{i_k} s_{i_{k+1}} = s_{i_{k+1}} s_{i_k}$

__Ex__: In type A_3



$s_2 s_1 s_3 s_2 s_1$

$\sim$



$s_2 s_3 s_1 s_2 s_3$

In type A, it's wiring diagrams where we don't care about order of commuting crossings

__Def__: (Stembridge)

$w \in W$ is called __fully commutative__ if all reduced decomp of w are in the same commutation class

__Lemma__: w is fully commutative iff $Inv(w)$ avoids root triple pattern 111

$\left\{ \begin{array}{c} \text{Fully commutative} \\ \text{elts in } W \end{array} \right\} \xleftrightarrow{\text{bij}} \left\{ \begin{array}{c} I \subset \Phi^+ \text{ avoiding} \\ 101, 010, 111 \end{array} \right\}$

In type A_{n-1} they are 321-avoiding perms in S_n

For other types,
We get a different generalization of Catalan #'s than for $\# J(\Phi^+)$