# 18.217 - Combinatorial Theory / 11.7.25

(Recall) an element $w \in W$ is <u>fully commutative</u> if all reduced decompositions are equivalent under commutation moves

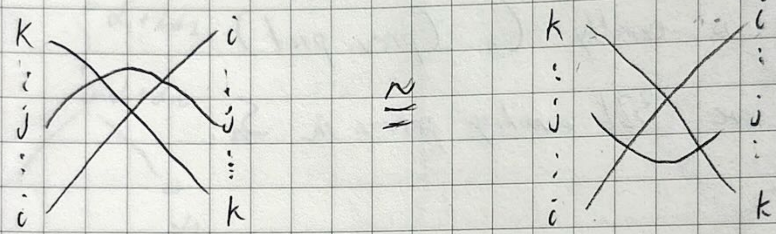
$$s_i s_j = s_j s_i$$

for $1 \le i < j \le l$. For type $A_{n-1}$, we have...

<u>Proposition:</u>

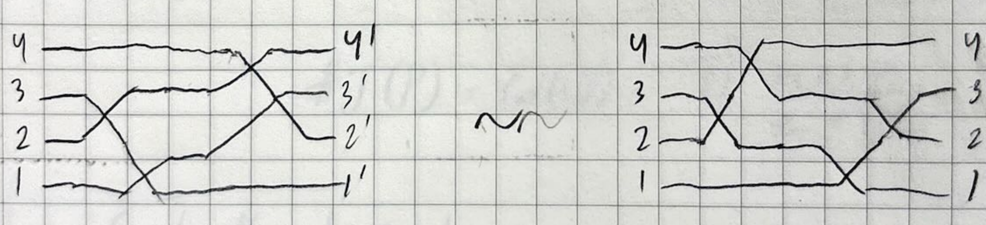
1) $w \in S_n$ is fully commutative
2) $w$ is 321-avoiding
3) all wiring diagrams for $w$ are equivalent (as graphs w/ labeled leaves, embedded into $\mathbb{R}^2$).

<u>Ex (wiring diagram)</u>:




this (3) implies (2), that $w(i) > w(j) > w(k)$. (equiv)

<u>Ex:</u>



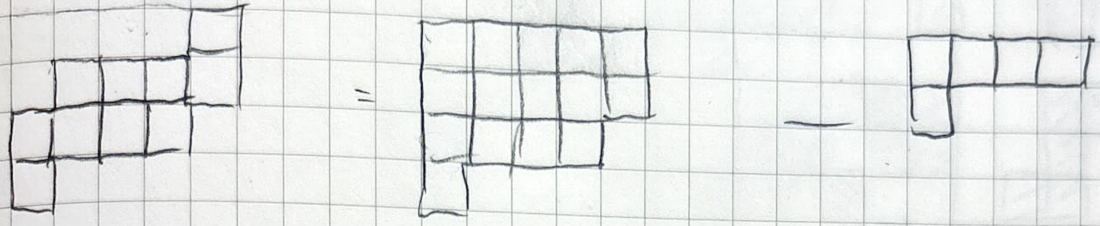

What can we say about wiring diagrams of fully commutative elements?

→ they correspond to <u>skew Young diagrams</u>

## Def'n (Skew Young Diagrams):

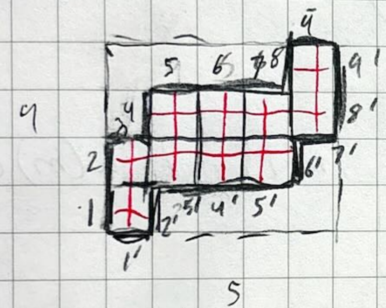
↳ Set theoretic difference of two Young diagrams

__Ex:__ consider $\lambda/\mu = (5,5,4,1)/(4,1)$



$\lambda/\mu$ $\quad\quad\quad$ $\lambda$ $\quad\quad\quad$ $\mu$

Notice that both $\lambda$ and $\mu$ fit in a $4\times 5$ grid, where the perimeter corresponds to a __lattice__ path on the grid.



→ gives a __fully commutative__ $w \in S_n$.

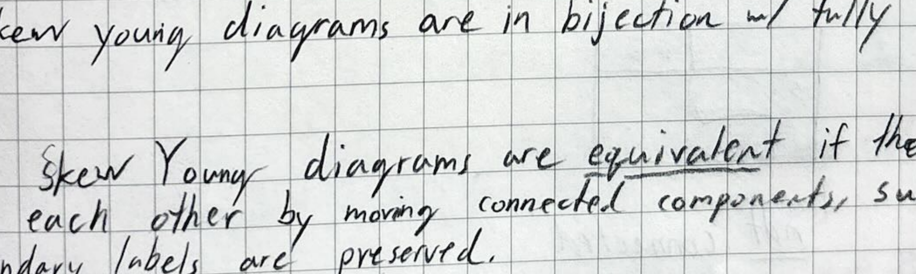
Namely,

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 6 & 1 & 8 & 3 & 4 & 5 & 9 & 7 \end{pmatrix}$$

__Claim:__ skew young diagrams are in bijection w/ fully commutative $w \in S_n$.

*__Note:__ two Skew Young diagrams are __equivalent__ if they are obtained from each other by moving connected components, such that the boundary labels are preserved.

__Ex:__ consider $\lambda/\mu = (4,3,2,1)/(2,2,2) \subseteq 5\times 4$ rectangle

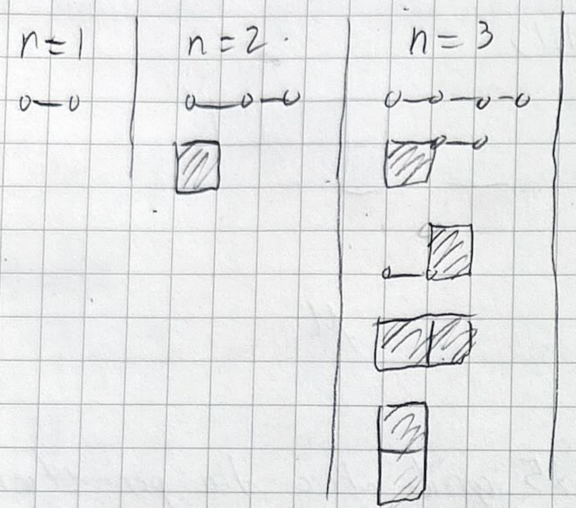


which is equivalent to $\lambda'/\mu' = (6$

→

(the bijection between Skew Young Diagrams and fully commutative $w \in S_n$ will appear on the next problem set.)

We can represent how many Skew Young Diagrams fit in an $n \times n$ square as




There are $C_n$ many distinct skew Young diagrams for each $n \times n$ square

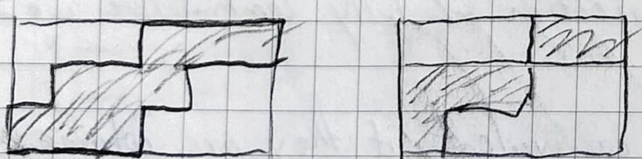
Def'n:

↳ $w \in S_n$ is <u>decomposable</u> if $\exists\, m \in \{1, \ldots, n-1\}$ s.t. $w(m) \in \{1, \ldots, m\}$ and $w(m+1), \ldots, w(n) \in \{m+1, \ldots, n\}$.

Otherwise $w$ is <u>indecomposable</u>

Def'n

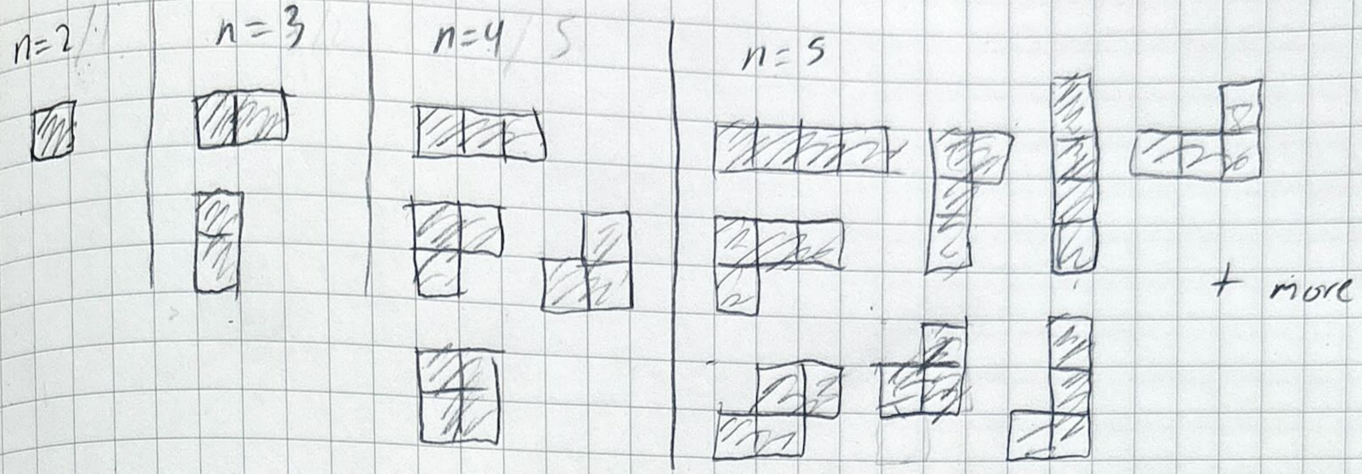
↳ A skew Young diagram $\lambda / \mu \subseteq k \times (n-k)$ rectangle is <u>connected</u> if each of it's $n-1$ diagonals are nonempty.



connected  <u>not</u> connected

<u>Exercise</u>: show that indecomposable fully commutative elements in $S_n$ are in bijection w/ connected skew diagrams $\lambda / \mu \subseteq k \times (n-k)$, for some $k \in \{1, 2, \ldots, n-1\}$.

How many indecomposable fully commutative elts are in $S_n$?




there are $2^{n-2}$ ribbons and $C_n$ many fully commutative elts in $S_n$.

Exercise: construct a bijection between all fully commutative elts. of $S_n$ and all indecomposable elts of $S_{n+1}$.

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We can now consider refinements of the Catalan numbers

→ Narayama Numbers $N(n,k) := \#$ Dyck paths w/ $k$ peaks

Thm: $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$.

We have another statistic on $S_n$, as well

→ Exceedence Number of $w$: $exc(w) = \{ i \in [n] \mid w(i) > i \}$.

Exercise: For $n \geq k \geq 1$, we have

1) $N(n,k) = \#$ of fully commutative elts in $S_n$ with $exc(w) = k-1$

2) $N(n,k) = \#$ of indecomp fully commutative elts in $S_{n+1}$ w/ $k$ exceedances. ($exc(w) = k$).

Exercise: Let $G$ be a simple graph with vertex set $[r]$. Then $G$-Temperly Lieb algebra (over $\mathbb{C}$ — or any field) with param $q \in \mathbb{C}$

generators: $e_1, e_2, \ldots, e_r$

relations: $e_i^2 = q e_i$, $e_i e_j e_i = e_i$ if $\{i,j\} \in G$. (and $e_i e_j = e_j e_i$

→ what is the dimension of this algebra?

→ how can we classify all corresponding graphs?