LECTURE 28 Wed 11/12

Def:
Temperley Lieb Algebra

$\mathbb{F}$ a field (e.g. $\mathbb{F} = \mathbb{C}$). $\delta \in \mathbb{F}$ a parameter

$TL_n(\delta)$ algebra (over $\mathbb{F}$)

generators: $e_1, e_2, \ldots, e_{n-1}$

relations: (1) $e_i^2 = \delta e_i$

(2) $e_{i+1} e_i e_{i+1} = e_{i+1}$

(3) $e_i e_j = e_j e_i$ for $|i-j| \geq 2$

An even more general algebra:

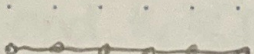
$G = (E, V)$ a simple graph on $V = [r]$

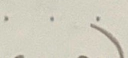
$TL_G(\delta)$: generators $e_1, \ldots, e_r$

relations: (1) $e_i^2 = \delta e_i$

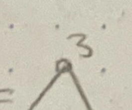
(2) $e_i e_j e_i = e_i$ $\forall$ edge $\{i,j\} \in E$

(3) $e_i e_j = e_j e_i$ $\forall$ $\{i,j\} \notin E$

Then $TL_n(\delta) = TL_G(\delta)$ for $G =$ 

Ex. 1) TL_3 a lin. basis: ($G =$ )

$1, e_1, e_2, e_1e_2, e_2e_1$

Ex. 2) TL_G for $G =$ 

A lin. basis $1, e_1, e_2, e_3, e_1e_2, e_2e_1, e_1e_3, e_3e_1, e_2e_3, e_3e_2$

$e_1e_2e_3, e_1e_2e_3e_1, e_1e_2e_3e_1e_2, \ldots$

This algebra is infinite dimensional!

Def: G is of finite TL-type if $TL_G(\delta)$ is finite dim'l.

Pset
Problem: Prove G is of finite TL-type $\Leftrightarrow$ G of finite coxeter type.

$\left\{ \begin{matrix} \text{fully commutative} \\ w \in S_n \end{matrix} \right\} \longrightarrow TL_n$

$w = s_{i_1} s_{i_2} \cdots s_{i_\ell} \longmapsto e_w = e_{i_1} e_{i_2} \cdots e_{i_\ell}$

Up to commutation, all reduced decomps are the same.

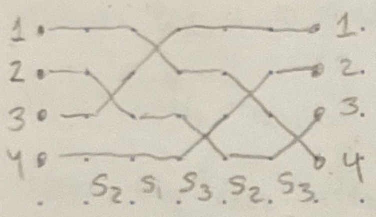
Claim: This map is well defined.

Thrm: $\{e_w \mid w \in S_n \text{ is fully commutative}\}$ is a linear basis of TL_n

equiv. 321-avoiding $\qquad \dim(TL_n) = C_n$

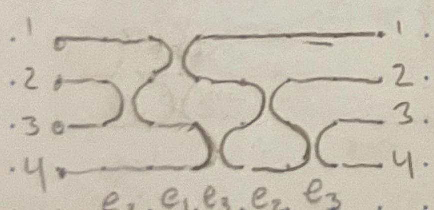
TL-wiring diagrams

Ex. $w = s_2 s_1 s_3 s_2 s_3$ a fully comm. elt. in S_4

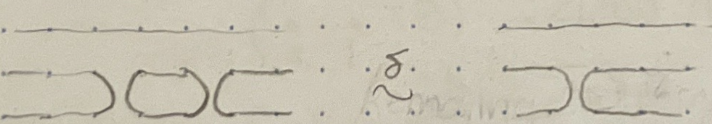


$w = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$

⇝ TL-wiring diagram: Replace each X with)(

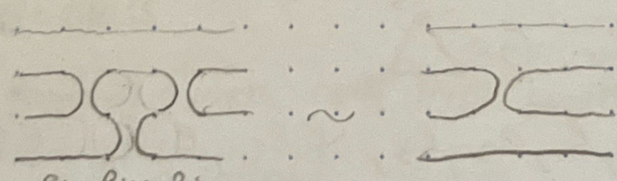


Relations: (1) $e_i^2 = \delta e_i$

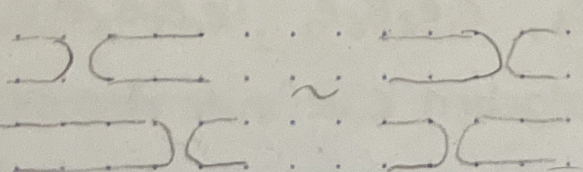


Removing closed loop gives a factor of δ

(2) $e_i e_{i+1} e_i = e_i$



(3) $e_i e_j = e_j e_i$
$|i-j| \geq 2$



Thrm: $\forall$ 2 sequences of indices, we have

$e_{i_1} e_{i_2} \cdots e_{i_\ell} = \delta^a e_{j_1} e_{j_2} \cdots e_{j_k}$ iff

the corresponding TL-wiring diagrams (with removed closed loops) are topologically equiv, and

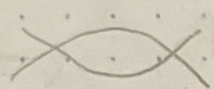
$a =$ (# closed loops in 1^{st} diagram) $-$ (# closed loops in 2^{nd} diagram)

Def: A decomp $e_{i_1} e_{i_2} \cdots e_{i_\ell}$ is TL-reduced if $\nexists$ shorter decomp s.t. $e_{i_1} \cdots e_{i_\ell} = \delta^a e_{j_1} \cdots e_{j_k}$ with $k < \ell$

Q: How can you tell from TL-wiring diagram if it's reduced

Cor: $e_{i_1} \cdots e_{i_\ell}$ is TL-reduced iff $s_{i_\ell} \cdots s_{i_1}$ is a reduced decomp in S_n of a fully commutative elt

↳ No strands double cross, no triple of strands all mutually cross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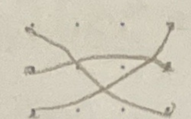
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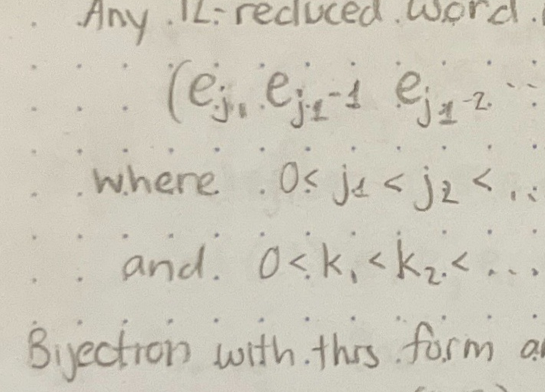
Thrm: (Jone's normal form)
Any TL-reduced word in TL_n can be uniquely rewritten in the form

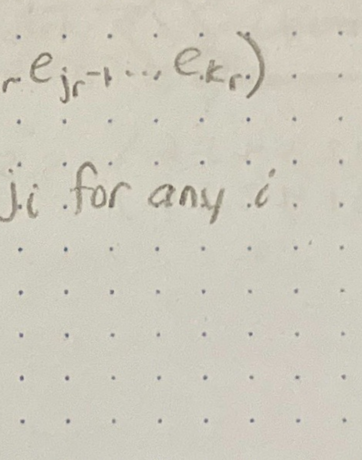
$$(e_{j_1} e_{j_1 - 1} e_{j_1 - 2} \cdots e_{k_1})(e_{j_2} e_{j_2 - 1} \cdots e_{k_2}) \cdots (e_{j_r} e_{j_r - 1} \cdots e_{k_r})$$

where $0 < j_1 < j_2 < \cdots < j_r < n$ and $k_i \le j_i$ for any i

and $0 < k_1 < k_2 < \cdots < k_r < n$.

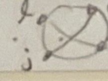
Bijection with this form and Dyck paths:





Pset Problem: Prove Jordan normal form

Cor: Basis elts $\{e_w \mid w \in S_n\}$ fall commutative can be labelled by non-crossing perfect matchings in K_{2n}. Ie. No edges (i,j) & (k,ℓ) s.t. $i<k<j<\ell$



Bijections:

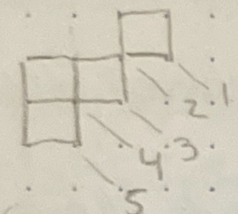
$\{\text{fully comm. elts in } S_n\} \longleftrightarrow \left\{\begin{array}{l}\text{Skew Young diagram w/ } n-1 \text{ diagonal} \\ \text{(possibly empty) labelled } 1, 2, \ldots, n-1\end{array}\right\}$

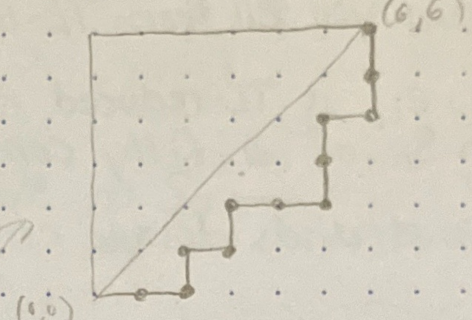
$\updownarrow$

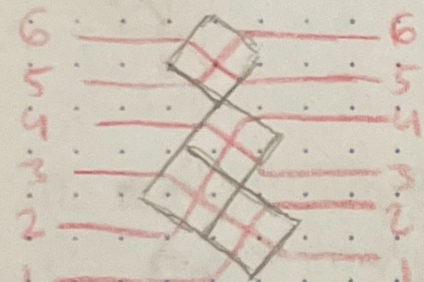
$\left\{\begin{array}{l}\text{Commutative classes of} \\ \text{reduced TL-wiring} \\ \text{diagrams for } TL_n\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Dyck paths w/ } 2n \text{ steps} \\ \text{(via Jones' form)}\end{array}\right\}$

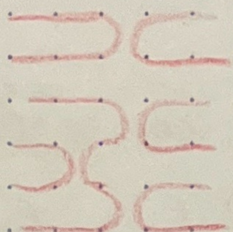
$\longleftrightarrow \left\{\begin{array}{c}\text{non-crossing perfect matchings} \\ \text{in } K_{2n}\end{array}\right\}$

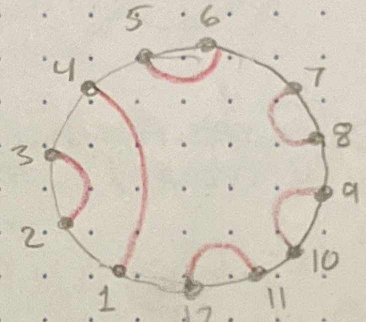
Ex. $n=6$

$\lambda/\mu =$ 









$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 2 & 6 & 5 \end{pmatrix} = s_2 s_1 s_3 s_5 \rightsquigarrow (e_2 e_1)(e_3)(e_5)$