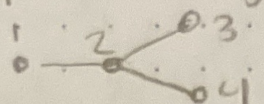
LECTURE 29 Fri 11/14

Recall

G-Temperly-Lieb Algebra $TL_G(\delta)$

Ex. $G=$ 

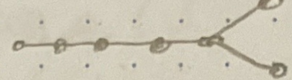
linear basis: $1, e_1, e_2, e_3, e_4$

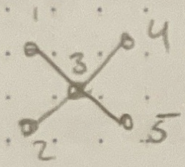
$e_i e_j$ $i \neq j$

$e_1 e_2 e_3 e_1 \ldots$ (no repeated indices)

$e_2 e_1 e_3 e_2$, $e_2 e_1 e_3 e_4 e_2$ etc

$$\boxed{\begin{array}{l}(1)\ e_i^2 \delta e_i \\ (2)\ e_i e_j e_i = e_i \text{ if } \{i,j\} \text{ an edge of } G \\ (3)\ e_i e_j = e_j e_i \text{ if } \{i,j\} \text{ is not an edge}\end{array}}$$

Exercise: Calculate $\dim TL_G$ for $G=$  (Type D_r)

Ex. $G=$  $\dim TL_G = \infty$

$e_3(e_1 e_2)e_3(e_4 e_5)e_3(e_1 e_2)e_3(e_4 e_5)\ldots$ etc

cannot write e_2 or e_1 b/c the could reduce it

Gives infinitely many different elts. of different lengths

A generalization: G is a directed graph

rel (2') $e_i e_j e_i = e_i$ if $i \longrightarrow j$

Ex. $TL_{1 \to 2}$ Basis: $1, e_1, e_2, e_1 e_2, e_2 e_1, e_2 e_1 e_2$

Even more general class of algebras

G directed graph

$\forall\ i \longrightarrow j$ $\quad m_{ij} \geq 3$, $m'_{ij} < m_{ij}$, $\varepsilon = \{1,2\}$

$$(2)''\ \underbrace{e_i e_j e_i e_j \ldots}_{m_{ij} \text{ things in product}} = \begin{cases} \underbrace{e_i e_j e_i \ldots}_{m'_{ij}} & \text{if } \varepsilon = 1 \\ \underbrace{e_j e_i e_j \ldots}_{m'_{ij}} & \text{if } \varepsilon = 2 \end{cases}$$

Symmetric Polynomials

$\Lambda_n = \mathbb{C}[x_1, \dots, x_n]^{S_n}$ symmetric polynomials in n vars

linear bases

(1) $m_\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$ + all permutations on indices
(Monomial symmetric basis).
For $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n)$ a partition w/ $\leq n$ parts

(2) $e_\lambda = e_1 e_2 \cdots e_{\lambda_\ell}$
(Elementary symmetric polynomials)
where $e_k := \sum_{1 \leq i_1 < \dots < i_k \leq n} x_{i_1} \cdots x_{i_k} = m_{(\underbrace{1,1,\dots,1}_{k},0,\dots,0)}$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ a partition with max part $\lambda_1 \leq n$

(3) $h_\lambda = h_1 h_2 \cdots h_{\lambda_\ell}$
(complete homogenous symm. poly.)
λ a partition with $\lambda_1 \leq n$

(4) Schur polynomials

$s_\lambda = \dfrac{a_{\lambda+\rho}}{a_\rho}$ where $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n)$ partition with $\leq n$ parts
$\rho = (n-1, n-2, \dots, 1, 0)$

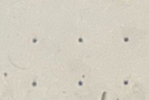
For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ (enough to assume $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$)

$a_\alpha = \sum_{w \in S_n} (-1)^{\ell(w)} x_{w(1)}^{\alpha_1} \cdots x_{w(n)}^{\alpha_n}$

(2), (3) $\Leftrightarrow$ Fundamental Thrm of symmetric poly.

$\Lambda_n \simeq \mathbb{C}[e_1, e_2, \dots, e_n] \simeq \mathbb{C}[h_1, h_2, \dots, h_n]$

<u>Thrm</u>: $s_\lambda = \sum_{\mu \leq \lambda} K_{\lambda\mu} m_\mu$



semi-standard young tableau

Where Kotska Number $K_{\lambda\mu}$ = #SSYT of shape λ & weight μ

Here "$\mu \leq \lambda$" is the dominance order $|\mu| = |\lambda|$
and $\mu_1 \leq \lambda_1$, $\mu_1 + \mu_2 \leq \lambda_1 + \lambda_2$, etc

We want to generalize this story to other types

Slight modification

$$\tilde{\Lambda}_n = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm n}]^{S_n}$$ ← symmetric Laurent Polynomials.

bases $\{m_\lambda\}$, $\{s_\lambda\}$ etc.

here $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n) \in \mathbb{Z}^n$

now can be negative

$$\tilde{\tilde{\Lambda}} = \left(\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/(x_1 x_2 \dots x_n - 1)\right)^{S_n}$$

bases $\{m_\lambda\}$, $\{S_\lambda\}$ etc.

$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n) \in \mathbb{Z}^n/(1, \dots, 1)\mathbb{Z}$

The set of integral weights in type A_n!

Generalization to other types

Φ crystallographic root system. W weyl group

P weight lattice. $P^{\geq 0} = P \cap C_0$ ← fundamental chamber

Group algebra of P

$\mathbb{C}[P]$ = the space of formal lin. comb. $\sum_{\lambda \in P} c_\lambda e^\lambda$ w/ fin. many terms

multiplication: $e^\lambda \cdot e^\mu = e^{\lambda+\mu}$

Note: $\mathbb{C}[P] \simeq \mathbb{C}[y_1^{\pm 1}, y_2^{\pm 1}, \dots, y_r^{\pm 1}]$ where $y_i = e^{\omega_i}$ ω_i fundamental weights

Rmk In Type A_{n-1} $x_i = y_i / y_{i-1}$ assuming $y_0 = y_n = 1$

$$\Lambda_W := \mathbb{C}[P]^W = \{f \in \mathbb{C}[P] \mid w(f) = f \ \forall\ w \in W\}$$

Linear bases of Λ_W.

Monomial Basis: $m_\lambda = \sum_{\substack{\text{distinct weights} \\ \mu = w(\lambda)}} e^\mu = \frac{1}{\#(\mathrm{Stab}\,\lambda)} \sum_{w \in W} e^{w(\lambda)}$

$= \#\{w \in W \mid w(\lambda) = \lambda\}$

Anologue of Schur functions:

(Characters of irred. representations of semisimple Lie algebras)

$$\chi_\lambda = \frac{a_{\lambda+\rho}}{a_\rho} \quad \text{where } \rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha = w_1 + w_2 + \dots + w_r$$

$$a_\alpha = \sum_{w\in W} (-1)^{\ell(w)} e^{w(\alpha)} \quad \text{antisymmetrizer } e^\alpha.$$

Exercise: Prove this. Can mostly follow same strategy as in type. The most non-trivial part is

<u>Lemma</u>: $a_\rho = e^\rho \prod_{\alpha\in\Phi^+} (1-e^{-\alpha})$

Generalized <u>Schur polynomials</u>

$$\chi_\lambda = \sum_{w\in W} w\left(\frac{e^\lambda}{\prod_{\alpha\in\Phi^+}(1-e^{-\alpha})}\right)$$

$G\subset\Phi^+$ a subset of positive roots.

<u>Def</u>: $S_\lambda^G = \sum_{w\in W}^{S_\lambda} w\left(\frac{e^\lambda}{\prod_{\alpha\in G}(1-e^{-\alpha})}\right)$

<u>Conj</u>. S_λ are non-neg. lin. comb. of $S_\mu^{(\alpha_1,\alpha_2,\dots,\alpha_r)}$

When you expand ones basis in terms of the other, seem to get whole numbers.

<u>Open Problem</u>: Find combinatorial Interpretation of these numbers