LECTURE 30 Mon 11/17

Schur polynomials for crystallographic root system Φ

$$S_\lambda = \frac{\sum_{w \in W} (-1)^{\ell(w)} w(e^{\lambda+\rho})}{\sum_{w \in W} (-1)^{\ell(w)} w(e^{\rho})} \in \mathbb{Z}[P]$$

$$= e^{\rho} \prod_{\alpha \in \Phi^+} (1-e^{-\alpha})$$

This is know as the Weyl Character Formula for characters $\chi(V_\lambda)$ of irreps of semi-simple Lie algebras.

$$= \sum_{w \in W} w\left(\frac{e^{\lambda}}{\prod_{\alpha \in \Phi^+} (1-e^{-\alpha})}\right)$$

Sum over lattice points μ of the permutohedron $\Pi(\lambda)$.

In type A_{n-1},

Kostka numbers

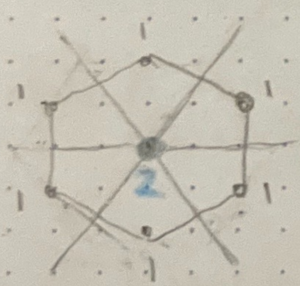
$$S_\lambda = \sum_{\mu} K_{\lambda,\mu} x^{\mu}$$

$\lambda, \mu \in \mathbb{Z}^n/(1,\dots,1)\mathbb{Z}$

$e^{\mu} = x^{\mu} = x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n} \quad (x_1 + \dots + x_n = 1)$

Ex. $n=3$

$A_2 \quad \lambda = (2,1,0)$



correspond to weights of the 6 verities

$$S_{(2,1,0)} = x_1^2 x_2^1 x_3^0 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2 + 2 x_1^1 x_2^1 x_3^1.$$

$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ # SYT of weight $(1,1,1)$ = weight of vertex $(1,1,1)$

Generalized Schur Polynomial

"Graph" $G \subset \Phi^+$

$$S_\lambda^G := \sum_{w \in W} w\left(\frac{e^{\lambda}}{\prod_{\alpha \in G} (1-e^{-\alpha})}\right)$$

In type A_{n-1}

$$s_\lambda^G = \sum_{w \in S_n} w\left(\frac{x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}}{\prod_{i<j} (1 - \frac{x_j}{x_i})}\right)$$

(i,j) is an edge of G

Thrm 1: $\forall G \subset \Phi^+$,

$\{s_\lambda^G \mid \lambda \in P^{\geq 0}\}$ is a linear basis in the ring W-invariant $\mathbb{C}[P]^W$

$P \cap C_0$

weight lattice

fundamental chamber

Proof Idea:

$$s_\lambda^G = \sum_{w \in W} w\left(\frac{e^\lambda \prod_{\alpha \in \Phi^+ \setminus G} (1 - e^{-\alpha})}{\prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})}\right)$$

$$= \sum_\nu \pm s_\nu$$

sum of all terms in the expansion of $e^\lambda \prod_{\alpha \in \Phi^+ \setminus G} (1 - e^{-\alpha})$

$\nu \leq \lambda$ ← Technically need to show still in chamber C_0

Thrm 2: $G = \{\alpha_1, \ldots, \alpha_r\}$ all simple roots.

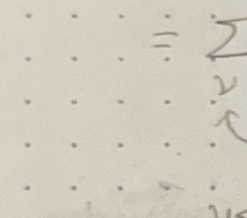
$$s_\lambda^G = s_\lambda^{fl} = \sum_{\substack{\text{all integer} \\ \text{lattice points } \mu \\ \text{of } \Pi(\lambda)}} e^\mu$$

← Take all non-zero terms from s_λ but set coeff to 1.

"flattened Schur polynomial"

Ex. $n=3$

A_2



$$x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2$$

$$+ 1 x_1 x_2 x_3$$

In type A_{n-1}, $s_\lambda^{fl} = \sum_{w \in S_n} w\left(\frac{x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}}{\prod_{i=1}^{n-1} (1 - \frac{x_{i+1}}{x_i})}\right)$

Open Problem: Describe all pairs of graphs $G \subset H \subset K_n$ s.t.

$s_\lambda^G = \sum_\mu c_{\lambda\mu} s_\mu^H$ where all $c_{\lambda\mu} \geq 0$.

Want to describe class of removable edges for G to get H.

Conj: If $G = K_n$ and $H = \underset{1}{\circ} \to \underset{2}{\circ} \to \underset{3}{\circ} \to \cdots \to \underset{n}{\circ}$

then we get a pair of basis with all non-neg. $c_{\lambda\mu}$.

(i.e. Schur polynomials are flattened-Schur positive)

Ex. $S_{(2,1,0)} = S^{fl}_{2,1,0} + S^{fl}_{1,1,1}$

Exercise: Type A_2

$$S_{(\lambda_1,\lambda_2,\lambda_3)} = \sum_{\substack{k \geq 0 \\ \lambda_1 - k \geq \lambda_2 \geq \lambda_3 + k}} S^{fl}_{(\lambda_1 - k,\, \lambda_2,\, \lambda_3 + k)}$$

Thm 2 follows from Brion's Formula

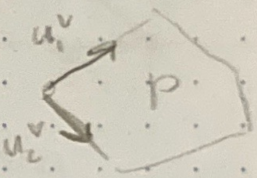
Brions Formula:

Let $P \subset \mathbb{R}^n$ be a convex polytope s.t.

(1) all vertices of P belong to $\mathbb{Z}^n$

(2) P is a simple n-dim'l. polytope
(Every vertex v is incident to exactly n edges)

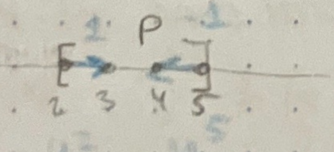
(3) $\forall$ vertices v, let $u_1^v, \ldots, u_n^v$ be the integer generators of the edges adjacent to v.
Then $u_1^v, \ldots, u_n^v$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^n$



Then $$\sum_{a \in P \cap \mathbb{Z}^n} x^a = \sum_{\substack{v \text{ vertex} \\ \text{of } P}} \frac{x^v}{\prod_{i=1}^{n} (1 - x^{u_i^v})}$$

Rmk: If P = permutohedren then this becomes exactly Thrm 2.

Ex. $n = 1$



$$x^2 + x^3 + x^4 + x^5 = \frac{x^2}{(1-x)} + \frac{x^5}{(1-x^{-1})}$$

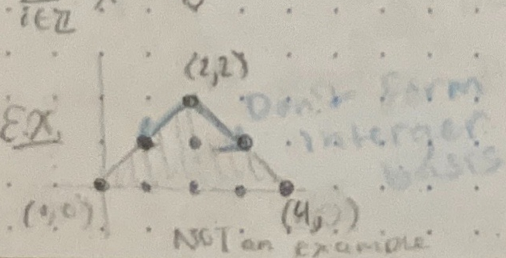
$$(x^2 + x^3 + x^4 + \ldots) + (x^5 + x^4 + x^3 + \ldots)$$
$$= (x^2 + x^3 + x^4 + x^5) + (\ldots + x^{-2} + x^{-1} + x^0 + x^1 + x^2 + \ldots)$$

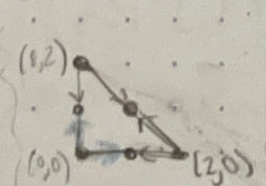
Key Identity

"$\sum_{i \in \mathbb{Z}} x^i = 0$"

Why: $\sum_{i \in \mathbb{Z}} x^i = \sum_{i \geq 0} x^i + \sum_{j < 0} x^{-j} = \frac{1}{1-x} + \frac{x^{-1}}{1-x^{-1}} = 0$

Ex.





$$1 + x_1 + x_1^2 + x_2 + x_1x_2 + x_2^2$$
$$= \frac{1}{(1-x_1)(1-x_2)} + \frac{x_1^2}{(1-x_1^{-1})(1-\frac{x_2}{x_1})} + \frac{x_2^2}{(1-x_2^{-1})(1-\frac{x_1}{x_2})}$$