LECTURE 31 Wed 11/19

Brion's Formula

$P \subset \mathbb{R}^n$ n-dim'l polytope satisfying the following.

(1) All vertices $\in \mathbb{Z}^n$

There's a version where you don't have to assume this, but then it gets a lot more technical, so we'll stick w/ this assumption.

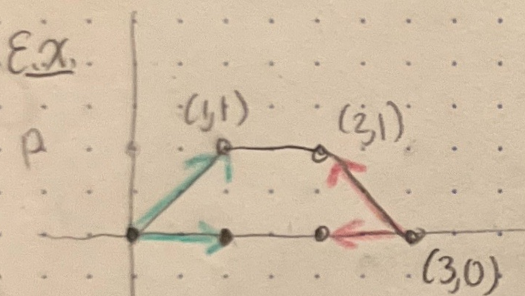
(2) P is simple (all vertices have same # of edges)

(3) The edges incident to a vertex v are generated by vectors $u_1^v, u_2^v, \ldots, u_n^v \in \mathbb{Z}_n$ s.t. $\det(u_1^v, \ldots, u_n^v) = \pm 1$ $\forall$ vertices v.
(they form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$)

Then $$\sum_{a \in P \cap \mathbb{Z}^n} x^a = \sum_{\substack{v \text{ vertex} \\ \text{of } P}} \frac{x^v}{\prod_{i=1}^n (1 - x^{u_i^v})}$$

$x^{a_1} x^{a_2} \ldots x^{a_n}$

Ex.



LHS: $1 + x_1 + x_1^2 + x_1^3$ ← bottom row y=0

$+ x_1 x_2 + x_1^2 x^2$ ← top row y=1

RHS: $\frac{1}{(1-x_1)(1-x_1x_2)} + \frac{x_1^3}{(1-x_1^{-1})(1-x_2/x_1)}$ ← bottom row vertices

$+ \frac{x_1x_2}{(1-x_1^{-1}x_2^{-1})(1-x_1)} + \frac{x_1^2 x_2}{(1-x_1^{-1})(1-x_y x_2)}$ ← top row vertices

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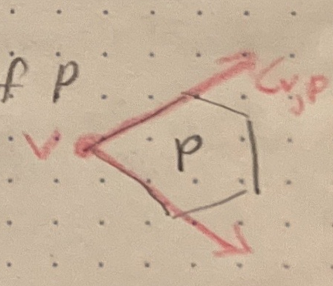
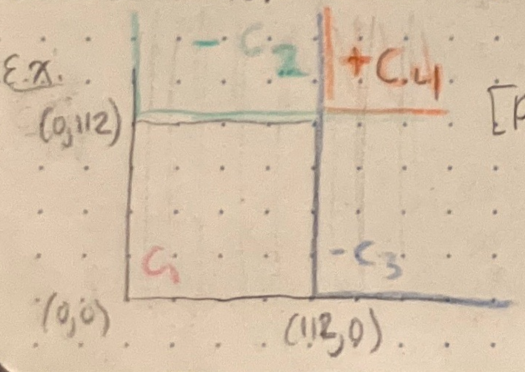
For $A \subset \mathbb{R}^n$, let $[A] := \sum_{a \in A \cap \mathbb{Z}^n} x^a$

For a vertex v of P, let $C_{v,P}$ be the "local cone" of P at vertex v generated by u_i^v.



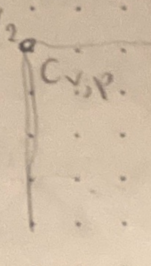
Then Brions Formula becomes $[P] = \sum_{v \text{ vertex}} [C_{v,P}]$

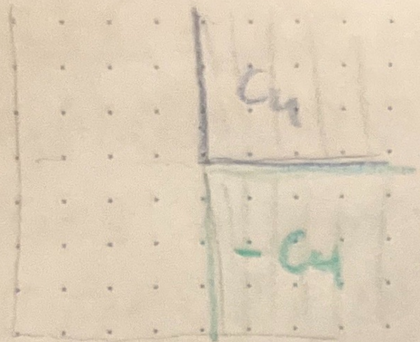
Ex.



$[P] = [C_1] - [C_2] - [C_3] + [C_4]$

similar picture at other vertices





$-[C_i] = [C_{v,p}]$

if you flip it, the sign changes

Key Identity

$$\sum_{i\in\mathbb{Z}} x^i = \sum_{i\geq 0} x^i + \sum_{i\leq 0} x^i$$
$$= \frac{1}{1-x} + \frac{x^{-1}}{1-x^{-1}} = 0$$

Space S of rational polyhedra in $\mathbb{R}^n$

Solution to system of linear eq. w/ rational coeffs

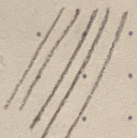
$P \rightsquigarrow$ characteristic function $\chi_P : \mathbb{R}^n \to \mathbb{R}$

$$\chi_P : x \mapsto \begin{cases} 1 & \text{if } x\in P \\ 0 & \text{if } x\notin P \end{cases}$$

S = linear space of fin. lin. combinations of χ_P's

Def: A rational polyhedron is ruled if it is a union of parallel lines.

Ex. P = half plane of $\mathbb{R}^2$



ruled

not ruled

Ruled $\iff$ has no vertex

A ruled polyhedron cannot be a polytope

bounded

Let $S' \subset S$ be the subspace generated by χ_P for ruled polyhedra

Thm: $\forall$ polytope P,

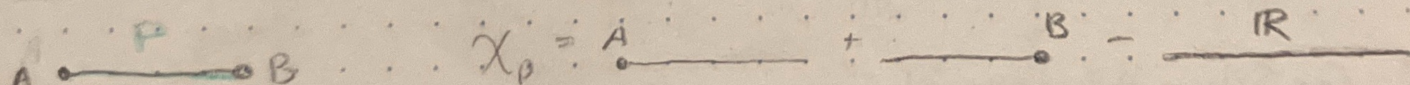
$$\chi_P \equiv \sum_{\substack{v \text{ vertex} \\ \text{of } P}} \chi_{C_{v,P}} \pmod{S'}$$ an identity is S/S'

Rmk

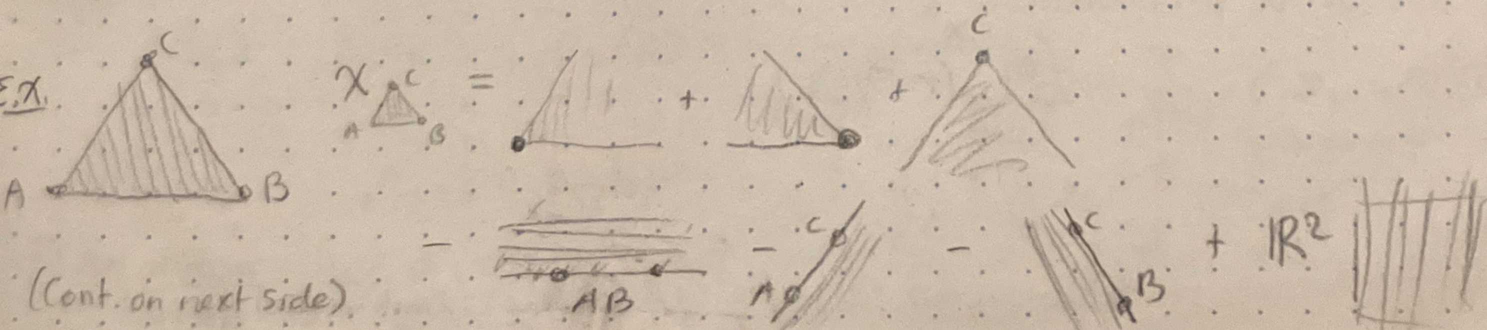
Want ruled line  to be generated by rational vector.

$\rightsquigarrow$ This is the one place where rationality is important

Ex.



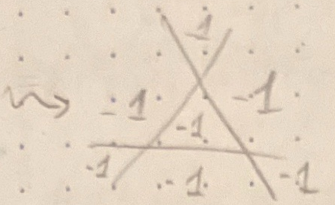
Ex.



(Cont. on next side)

To keep track

(pencil lines from adding cones)
Colored pencil - subtracting half planes

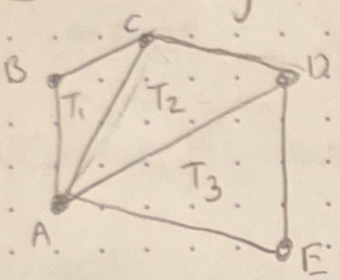


Exercise: Generalize this to an n-dim'l simplex

Claim: The case when P is a simplex implies the general case of Thrm.

Proof sketch: Triangulate P.

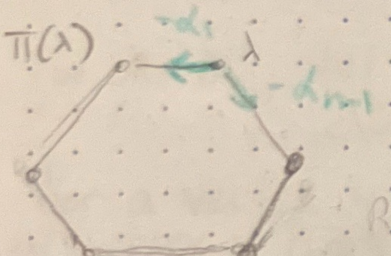
P =



$$\chi_P = \chi_{T_1} + \chi_{T_2} + \chi_{T_3} - \chi_{[A,C]} - \chi_{[D,C]}$$

Apply Thrm to each simplex.

Assume $P = \Pi(\lambda) = \mathrm{conv}(w(\lambda) \mid w \in S_n)$



Generators are -(simple roots)

$\alpha_i = (0, \dots, 0; \underset{i}{1}, \underset{i+1}{-1}, 0, \dots, 0)$ simple root

Recall: Flattened Schur polynomial

$$s_\lambda^{fl}(x_1, \dots, x_n) = \sum_{\mu \in \Pi(\lambda) \cap \mathbb{Z}^n} x^\mu \overset{\text{Brion}}{=} \sum_{\substack{w \in S_n \\ \uparrow \\ \text{vertices}}} w\left(\frac{x^\lambda}{\prod_{i=1}^{n-1}\left(1 - \frac{x_{i+1}}{x_i}\right)}\right)$$

Def: G-Schur polynomial

$$s_\lambda^G(x_1, \dots, x_n) = \sum_{w \in S_n} w\left(\frac{x^\lambda}{\prod_{\substack{1 \le i < j \le n \\ (i,j) \in G}}\left(1 - \frac{x_j}{x_i}\right)}\right)$$

Next goal: To understand

$$\frac{1}{\prod_{\substack{1 \le i < j \le n \\ (i,j) \in G}}\left(1 - \frac{x_j}{x_i}\right)} = \sum_\mu P_G(\mu)\, x^{-\mu}$$

Restricted Kostant's Partition Function