LECTURE 32 Fri 11/21

(Restricted) Kostant's Partition Function $p_G(\beta)$

$G \subseteq \Phi^+$ a "graph"

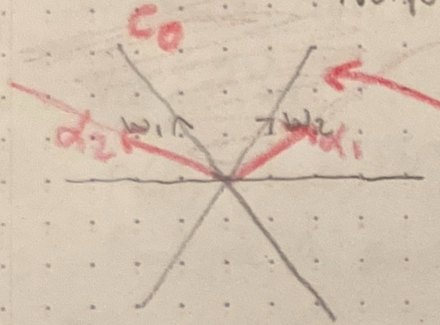
$\prod_{\alpha \in G} \frac{1}{1-e^\alpha} = \sum_\beta p_G(\beta) e^\beta$ ← Over all β, but it's 0 unless β is an integer lattice point of the dominant cone (i.e. $\beta \in C_0 \cap \mathbb{Z}^r$)

$= \prod_\alpha (1 + e^\alpha + e^{2\alpha} + \dots)$

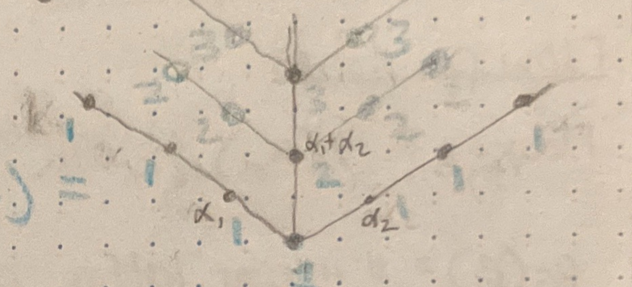
Note: The usual (unrestricted) Kostant's partition function is $p(\beta) = p_{\Phi^+}(\beta)$.

Def: $p_G(\beta) = \#\{(c_\alpha)_{\alpha \in G} \in \mathbb{Z}_{\geq 0}^{|G|} \mid \beta = \sum_{\alpha \in G} c_\alpha \alpha\}$

= Ways to express β as a non-neg. linear combo of $\alpha \in G$



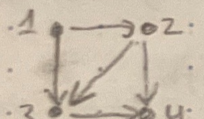
$p_B(G) =$



In type A_{n-1}, $G \subseteq K_n$. $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{Z}^n$ s.t. $\beta_1 \geq 0$, $\beta_1 + \beta_2 \geq 0$, $\dots$, $\beta_1 + \dots + \beta_n \geq 0$.

Edge (i,j) included if we have root $(0, \dots, 1, 0, \dots, 0, -1, 0)$.

Ex 1 $n=4$ (type A_3)

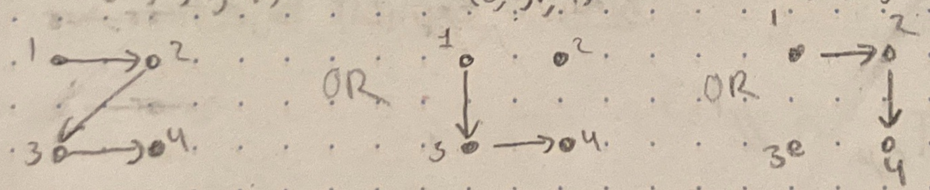
$G =$ 

$\beta = (1,0,0,-1)$

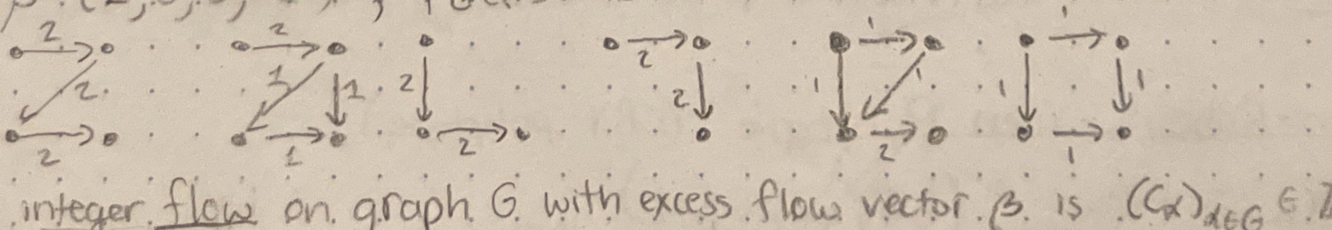
$p_G(\beta) = 3$

Can obtain β from graphs

$(1,0,0,-1) = (1,-1,0,0) + (0,1,-1,0) + (0,0,1,-1)$ OR $(1,0,-1,0) + (0,0,1,-1)$ OR $(1,-1,0,0) + (0,1,0,-1)$



If $\beta = (2,0,0,-2)$, $p_G(\beta) = 6$



Def: An integer flow on graph G with excess flow vector β is $(c_\alpha)_{\alpha \in G} \in \mathbb{Z}_{\geq 0}^{|G|}$ s.t. $\sum_{\alpha \in G} c_\alpha \alpha = \beta$.

$p_G(\beta) = \#$ integer flows on G with excess flow vector β.

(Generalized) Schur polynomials:

$$s_\lambda = \frac{\sum_{w\in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in\Phi^+} (1-e^{-\alpha})} = \sum_{w\in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\beta}$$

$$= \sum_{\beta} p(\beta) e^{-\beta}$$

$$s_\lambda^G = \frac{\text{Same numerator}}{\prod_{\alpha\in G} (1-e^{-\alpha})}$$

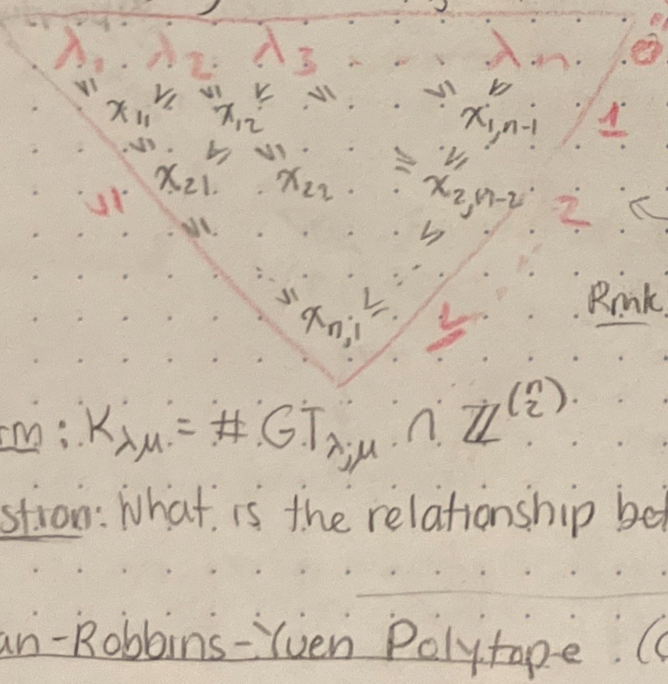
<u>Flow polytopes</u>

$$\mathrm{Flow}_{G,\beta} = \left\{ (x_\alpha)_{\alpha\in G} \in \mathbb{R}^{|G|} \;\middle|\; \begin{array}{l} x_\alpha \geq 0 \ \forall \alpha\in G, \\ \sum_{\alpha\in G} x_\alpha \cdot \alpha = \beta \end{array} \right\}$$

$p_G(\beta) = \#$ integer lattice points in $\mathrm{Flow}_{G,\beta}$

<u>Gelfand Tsetlin Polytope</u> $GT_{\lambda,\mu} \subset \mathbb{R}^{\binom{n}{2}}$ (for type A only)

Consists of all vectors $(x_{11}, x_{12}, \dots, x_{1,n-1}, x_{21}, x_{22}, \dots, x_{2,n-2}, \dots, x_{n1})$ satisfying $x_{ij} \in \mathbb{R}$



$\mu_i = (i+1)^{st}$ row sum $-$ i^{th} row sum for $i = 1, \dots, n$.

<u>Def</u>: Gelfand Tsetlin pattern for λ is shape like this w/ integer values

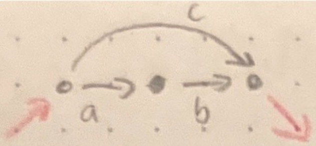
<u>Rmk</u>: $\exists$ explicit bijection w/ SSYT

<u>Thrm</u>: $K_{\lambda\mu} = \# GT_{\lambda,\mu} \cap \mathbb{Z}^{\binom{n}{2}}$

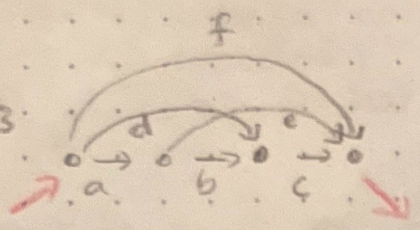
<u>Question</u>: What is the relationship between $G_{\lambda,\mu}$ and $\mathrm{Flow}_{K_n,\beta}$?

<u>Chan-Robbins-Yuen Polytope</u> (CRY-polytope)

$\mathrm{CRY}_n := \mathrm{Flow}_{K_{n+1}}(1, 0, \dots, 0, -1)$

Ex. $n=2$ $K_3 =$ 

$$\left\{\begin{array}{l} a, b, c \geq 0 \\ a+c=1 \quad a=b \\ b+c=1 \end{array}\right\} \Leftrightarrow \{0 \leq a \leq 1\}$$

CRY_3 

$$\left\{\begin{array}{l} a, b, \ldots, f \geq 0 \\ 1 = a+d+f \\ f+e+c=1 \\ a=b+e, \ b+d=c \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} x, y, z \geq 0 \\ x+y+z \leq 1 \end{array}\right\}$$

$x=a-b$; $y=c-b$; $z=b$

$CRY_3 =$ 

3D polytope
6 vars, 5 equations, but some are lin. dep.

In general $\dim CYR_n = \binom{n}{2}$

Vol. CRY_n = ?

$p(N, 0, \ldots, 0, -N) = ?N^{\binom{n}{2}} + \ldots$

↖ type A_n Kostant's formula

Thrm (Conj. by Chan-Robbins-Yuen, proved by Ziellberger)

$\binom{n}{2}!\ \mathrm{Vol}\ CRY_n = C_1 \cdot C_2 \cdot C_3 \cdots C_{n-2}$ ← Product of Catalan numbers

Thrm: $\binom{n}{2}\ \mathrm{Vol}\ CRY_n = P_{K_n}(1, 2, 3, \ldots, n-1, -\binom{n}{2})$

Volume of $\mathrm{Flow}_{K_{n+1}}(1, 0, \ldots, 0, -1)$ ↑

↖ # lattice points of different flow polytope

Open Problem(?): Provide a bijective proof of the above thrm.