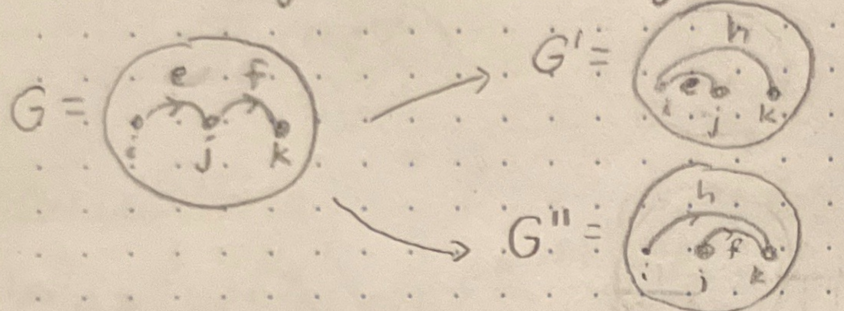
LECTURE 33 Mon 11/24

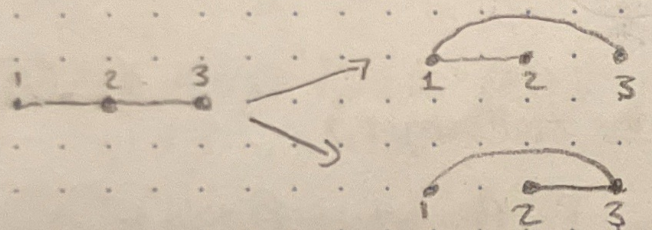
Elliott-MacMahan Game on Graphs

G a graph on vertices $1, 2, \ldots, n$, edges directed as $i \to j$ for $i<j$. Multiple edges are allowed, No loops.

Moves. If we can find two edges $i \xrightarrow{e} j \xrightarrow{f} k$ ($i < j < k$) in G then replace G by two graphs G' and G'' obtained from G by replacing one of these edges with $i \xrightarrow{h} k$.

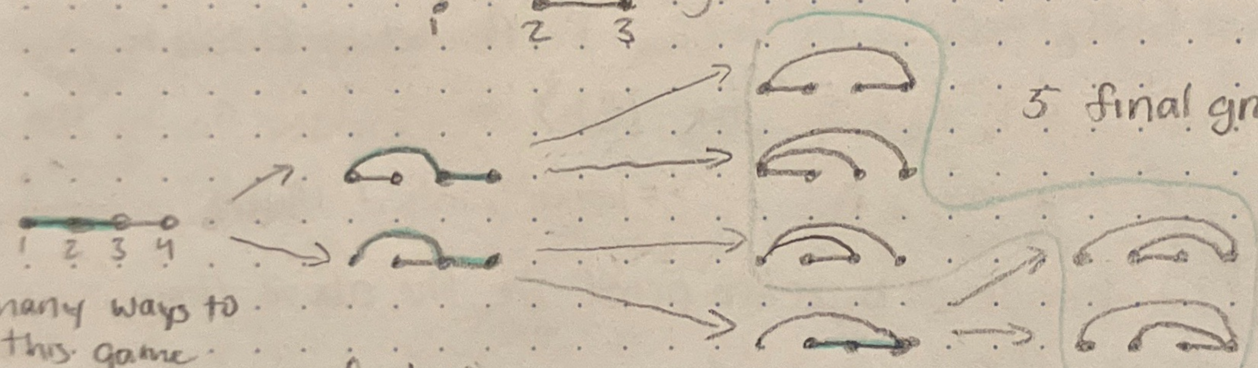


Ex.



2 final graphs

Also 2 for complete graph b/c 3rd edge just increases multiplicity



5 final graphs

How many ways to play this game (i.e. to pick which pair of edges to replace at each step)

Thm: 1.) Any way to play the game produces the same number f_G of final graphs

2.) For the chain $G = 1 - 2 - 3 - \cdots - n$, $f_G = C_{n-1}$ (the Katalan number).

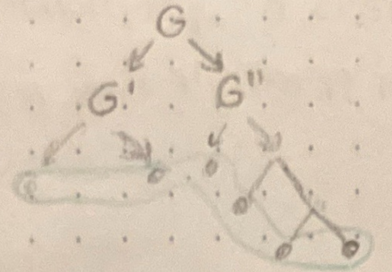
3.) For complete graph get series $1, 2, 2\cdot5, 2\cdot5\cdot14, \ldots$

$$f_{K_n} = C_1 \cdot C_2 \cdots C_{n-1}$$

This number briefly appeared at end of last lecture and that's not a coincidence

Exercise: Prove these.

<u>Clarification</u>: Think of game as constructing binary tree starting with G as the root. Number of graphs at the end = # leaves (and we only care about multiplicities of edges in G, the specific edges aren't labelled)



(Restricted) Kostant's partition function G.

$p_G(\beta)$ = # ways to express β as non-neg. linear combination

$\boxed{\beta=(\beta_1,\dots,\beta_n)\in\mathbb{Z}^n,\ \sum\beta_i=0}$

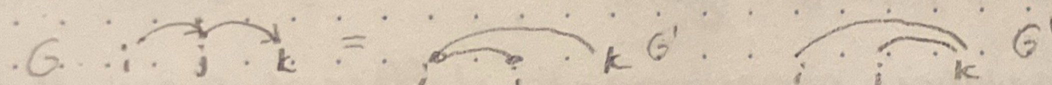
$$\beta = \sum_{\substack{\text{edge } i\to j\\ \text{in } G}} c_f(\vec{e}_i-\vec{e}_j), \quad c_f\in\mathbb{Z}_{\geq 0}$$

$$= [x_1^{\beta_1}\cdots x_n^{\beta_n}] \prod_{f:\, i\to j \text{ in } G} \frac{1}{(1-x_i/x_j)}$$

coeff of $x_1^{\beta_1}\cdots x_n^{\beta_n}$ in the product

$$= [x_1^{\beta_1}\cdots x_n^{\beta_n}] \prod_{i\to j \text{ in } G} \frac{x_1^{d_1}\cdots x_n^{d_n}}{x_j-x_i} \qquad d_i=\text{indeg}_G(i)$$

<u>Key Identity</u>: $$\frac{1}{(x_i-x_j)(x_j-x_k)} = \frac{1}{(x_i-x_k)(x_j-x_k)} + \frac{1}{(x_i-x_j)(x_i-x_k)}$$



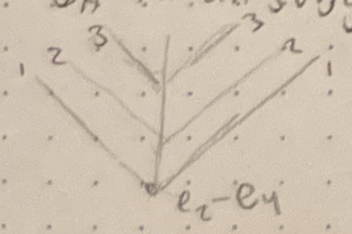
<u>Cor</u>: $p_G(\beta)=p_{G'}(\beta)+p_{G''}(\beta')$

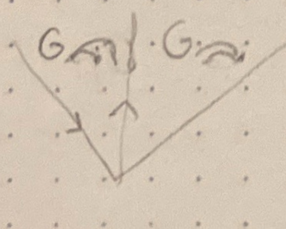
<u>Thrm</u>: $p_G(\beta)$ is a piecewise polynomial function on integer points β of the cone $\mathcal{C}_G$ = spanned by e_i-e_j for all edges $i\to j$ of G

In particular $p(\beta)=P_{K_n}(\beta)$ is a function on the dominant cone $\mathcal{C}_{K_n} = \left\{ \begin{matrix} \beta_1\geq 0;\ \beta_1+\beta_2\geq 0 \dots \\ \beta_1+\beta_2+\dots+\beta_n=0 \end{matrix} \right\}$

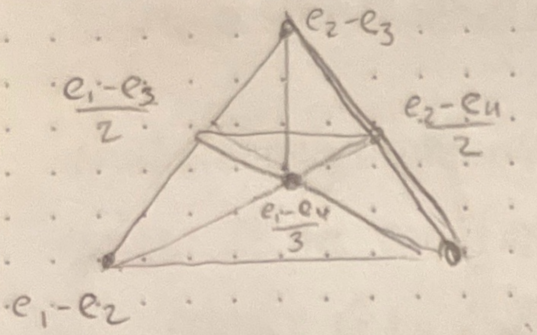
It's areas of polynomiality are the cones from the common refinement of $\mathcal{C}_G$ by cones of the form $\mathcal{C}_H$ for all subgraph $H\subseteq G$

<u>Ex.</u>, $n=3$





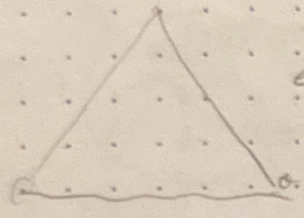
$n=4$ $p_4(\beta_1, \beta_2, \dots, \beta_4)$



7 areas of polynomiality

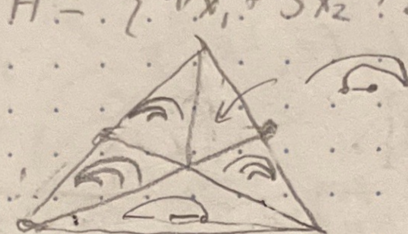
Each triangle here represents one 3D cone.

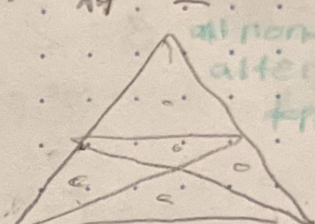
= The common refinement of 2 triangulations



$\leftarrow \mathcal{C}_4 \cap H = \{4x_1 + 3x_2 + 2x_3 + x_4 = 1$

Two different triangulations!





all non-nesting alternating trees

5 triangles in eah triangulation

all non-crossing alternating trees

In general: Look at simplex given by cone intersected w/ simplex whose vertices correspond to positive roots.

Mostly all results about these can be used by using induction and Elliott-Macmahan game

Different triangulation correspond to different ways to play the game