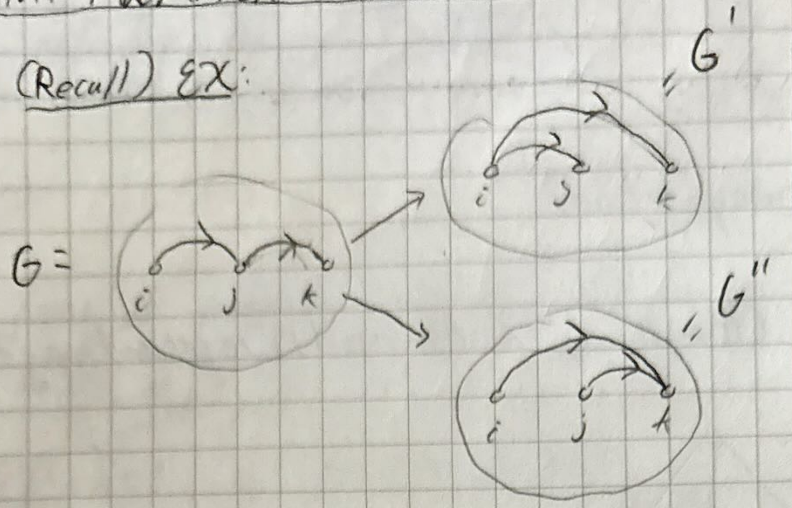
18.217 – Combinatorial Theory / 11.26.25

Elliott MacMahon Game (cont.)

(Recall) EX:



for $i<k<j$

$G = ([n], E)$ is a directed graph

each game produces a binary tree, with vertices labeled by graphs obtained after certain moves (described above).

Essentially given a directed graph G with $V(G) = [n]$, and directed edges $i \to j$, with $i<j$, and multiple edges allowed, we can take moves from subpaths $i \to j \to k$ as above, and continue playing until no possible moves remain. (no loops allowed)

Consider also

$$m_G = \prod_{(i,j)\in E(G)} \frac{1}{x_j - x_i}$$

and by our Key Lemma, we have $m_G = m_{G'} + m_{G''}$

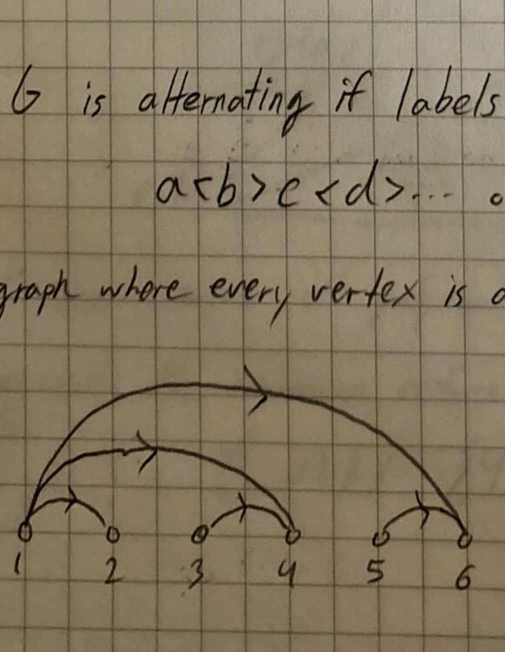
Def'n:

↳ a graph G is alternating if labels in any path alternate

$$a<b>c<d>\dots \quad \text{or} \quad a>b<c>d<\dots$$

(equiv: a graph where every vertex is a source or a sink)

EX:



} alternating tree

Corollary: $m_G = m_{G_1} + \cdots + m_{G_N}$, where G_i's are all final graphs, (resp. leaves of the corresponding binary tree) for any EM-game, starting with G.

This game is related to Kostant's partition function, $p_G(\beta)$, for $G = K_n$

$$\tilde{m}_G = \prod_{(i,j)\in E(G)} \frac{1}{1 - x_i/x_j}$$

$$= m_G \cdot \prod_i x_i^{\mathrm{indeg}_G(i)}$$

we can also write $\tilde{m}_G$ in terms of linear combinations of positive roots

$$\tilde{m}_G = \sum_{\substack{\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n \\ \text{w/ } \beta_1 \geq \beta_2 \geq \cdots \geq \beta_n \\ \sum \beta_i = 0}} p_G(\beta) \cdot x_1^{\beta_1} \cdots x_n^{\beta_n}$$

expansion of m_G as a power series of x_i that converge in the region $|x_1| \ll \cdots \ll |x_n|$

Lemma: $p_G(\beta) = [x_1^{\beta_1 - \mathrm{indeg}_G(1)}, \ldots, x_n^{\beta_n - \mathrm{indeg}_G(n)}](m_G)$

(Recall)

$$\frac{1}{x_j - x_i} = x_j^{-1}\left(\frac{1}{1 - x_i/x_j}\right) = \sum_{k\geq 0} \frac{x_i^k}{x_j^{k+1}} \quad \text{w/ } i<j$$

Corollary: $p_G(\beta) = p_{G'}(\beta) + p_{G''}(\beta + e_j - e_k)$

Lemma: for any tree T (w/ possibly multiple edges) $p_T(\beta)$ is a polynomial on lattice points of the cone $\mathcal{C}_T$, generated by roots $e_i - e_j$ for $(i,j) \in E(T)$ and 0 outside of $\mathcal{C}_T$.

Proof: let m_{ij} be the multiplicity of edges $i \to j$ in T. Now, let

$$\beta = \sum_{\substack{(i,j)\in E(T) \\ \text{w/ } i \to j}} c_{ij}(e_i - e_j)$$

then $\beta \in \mathcal{C}_T$ iff all $c_{ij} \geq 0$ and

polynomial in c_{ij}

$$\underset{\beta \in \mathcal{C}_T}{p_T(\beta)} = \prod_{(i,j)} \binom{c_{ij} - m_{ij} - 1}{m_{ij} - 1}$$

Then, one region of polynomiality is $p_G(\beta)$, where G is a graph on n vertices, $1,\dots,n$ and $\tilde{G}$ is the induced subgraph on $n-1$ vertices $1,2,\dots,n-1$.

Assume $\beta = (\beta_1,\dots,\beta_{n-1}, -(\beta_1+\dots+\beta_{n-1}))$, where $\beta_1,\dots,\beta_{n-1} \geq 0$. Let $d_i = \text{outdeg}_G(i)-1$, and assume $d_1,\dots,d_{n-1} \geq 0$.

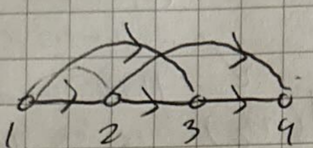
<u>Thm:</u> $$p_G(\beta) = \sum_{(v_1,\dots,v_{n-1})} p_{\tilde{G}}(v_1-d_1, v_2-d_2,\dots, v_{n-1}-d_{n-1}) \binom{\beta_1+d_1}{v_1}\cdots\binom{\beta_{n-1}+d_{n-1}}{v_{n-1}}$$

$v \in \mathbb{Z}^{n-1}$

$\forall k,\ \sum_{i=1}^{k}(v_i-d_i) \geq 0$ and $v_1+\dots+v_n = d_1+\dots+d_n$

*<u>Note:</u> if G has exactly one edge $(n-1)\to n$, then $d_{n-1}=0$, $v_{n-1}=0$

<u>Ex:</u> let G be ~~the~~ the following graph



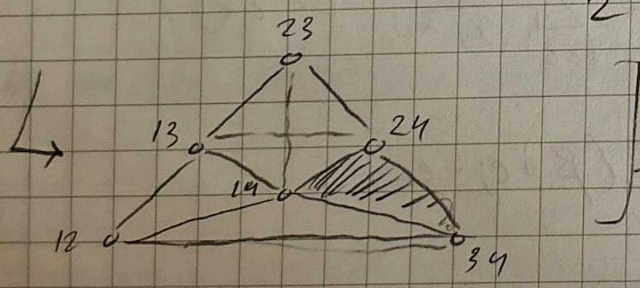
$d_1 = d_2 = 1,\ d_3 = 0$

$v = (2,0,0),\ (1,1,0)$

$$p_G(x,y,z,-x-y-z) = p_{\tilde{G}}(1,-1,0)\binom{x+1}{2} + p_{\tilde{G}}(0,0,0)\binom{x+1}{1}\binom{y+1}{1}$$

for $x,y,z \geq 0$

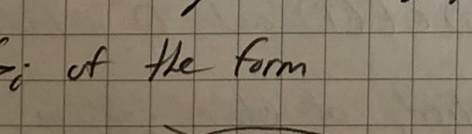
$$= \frac{x(x+1)}{2} + (x+1)(y+1)$$



Cone spanned by $e_1-e_2,\ e_2-e_3,\ e_3-e_4$

<u>Idea of proof:</u> play the EM-game on a graph G, to obtain $G_1,\dots,G_N$.

→ only graphs G_i of the form



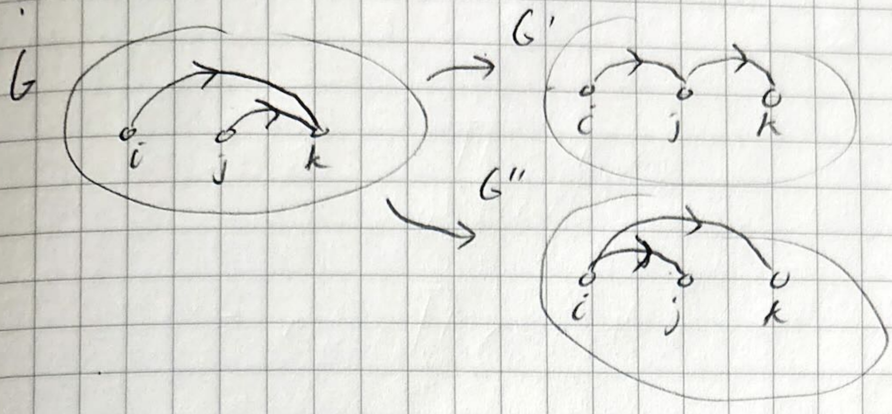
contribute (only edges from $i\to n$) → out-deg. of vertices

↳ let $u_i = \text{outdeg}_T(i)-1$

→ # of times a tree T appears among G_i's $= p_{\tilde{G}}(v-\tilde{d})$.

This does ~~not~~ not prove piecewise polynomiality, as we could get an alt. graph that is not a tree.

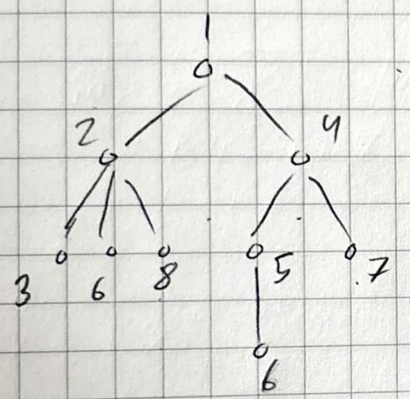
To finish this up, we need to play another game on graphs



$m_G = m_{G'} - m_{G''}$

w/ m_G as the sum of m_{G_i} for graphs w/ a pair of edges

equivalently, this is an alternating sum over increasing trees (w/ mult. edges)



$\Rightarrow$ piecewise polynomial of $p_G(\beta)$