THE CARTAN-LERAY SPECTRAL SEQUENCE OF THE BRAID GROUP (REVISION)

DEZHOU LI

Abstract. In Cohen's famous calculation of the mod p cohomology of configuration spaces, the key ingredient was a complete description of the Cartan–Leray spectral sequence for the configuration space of $k = p$ points. I will discuss this aimed at giving a complete description of this spectral sequence for arbitrary k. This work not only provides a geometric way to prove the Arone–Mahowald theorem and Kjaer's theorem, but also gives the potential to determine the ring structure of cohomology of unordered configuration spaces.

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1. Introduction

Configuration spaces lie at the intersection of fields as diverse as iterated loop space theory, knot theory, number theory, and mathematical physics.

Definition 1.1. The *configuration space* of k ordered points in the topological space X is

$$\mathrm{Conf}_k(X) := \{(x_1, x_2, \dots, x_k) \in X^k : x_i \neq x_j \text{ if } i \neq j\}$$

endowed the subspace topology of X^k. The *unordered configuration space* is the quotient

$$B_k(X) := \mathrm{Conf}_k(X)/\Sigma_k.$$

For the case of $X = \mathbb{R}^2$, $B_k(\mathbb{R}^2)$ is the classifying space of the braid group. Since the action Σ_k on $\mathrm{Conf}_k(\mathbb{R}^n)$ is free, so for each k and n, there is a Cartan–Leray spectral sequence of the form

$$E_2^{s,t} \cong H^s\Big(\Sigma_k, H^t(\mathrm{Conf}_k(\mathbb{R}^n); \mathbb{F}_p)\Big) \Longrightarrow H^{s+t}(B_k(\mathbb{R}^n); \mathbb{F}_p).$$

In Cohen's famous calculation [FM76] of the mod p cohomology of configuration spaces over $\mathbb{R}^n$, the key ingredient was a complete description of the Cartan–Leray spectral sequence for the configuration space of $k = p$ points. As such, giving a complete description of this spectral sequence mod p for arbitrary k is a natural question to ask. Formally,

Problem 1.2. For every k and n, determine the Cartan–Leray spectral sequence for the action of Σ_k on $\mathrm{Conf}_k(\mathbb{R}^n)$ mod p.

There are several applications of this problem. First, we recall that the transfer maps for the coordinate projections endow

$$A_n := \bigoplus_{k>0} H^*(B_k(\mathbb{R}^n); \mathbb{F}_p)$$

with the structure of a divided power algebra [GSS23].

Application 1.3. Give a complete description of A_n as a divided power algebra.

Next, recall from above that the entries in the top row of the spectral sequence are of the form $H^*(\Sigma_k; M_k)$, where

$$M_k := H^{(k-1)(n-1)}(\mathrm{Conf}_k(\mathbb{R}^n); \mathbb{F}_p).$$

These vector spaces may be interpreted as certain spaces of Lie algebra power operations [Chi05], and a difficult calculation of Arone–Mahowald shows that all such operations vanish for $k = p^r$ or $k = 2p^r$ for $r \geq 1$ [AM99].

Application 1.4. Give a geometric proof of the Arone–Mahowald theorem.

Kjaer's theorem from [Kja17] shows that among the cases where the top row does not vanish, it is possible to calculate its dimension over $\mathbb{F}_p$.

Application 1.5. Give a geometric proof of Kjaer's theorem.

Note that the same technique can be applied to study the general $B_k(\mathbb{R}^n)$. Besides, since the cup product is compatible with the spectral sequence, this creates a potential to study its ring structure, which is still an open question until these days. Moreover, we can incorporate Hopf ring [GSS23] and/or operad [May72] structure into the Cartan–Leray spectral sequence description, which brings extra structures on A_n.

For this paper, we have been able to solve the main problem up to the case $k \leq 2p^2$. We will phrase our answer in terms of certain atomic spectral sequences $U_{(m,n,r)}$ and $V_{(m,n)}$.

Definition 1.6. Let m, n and r be integers and $0 \leq m \leq n$, $r \geq 2$. We define $U_{(m,n,r)}$ to be a first quadrant spectral sequence over $\mathbb{F}_p$ such that

$$d_r^{m,n-m} : E_r^{m,n-m} \cong \mathbb{F}_p \to E_r^{m+r,n-m-r+1} \cong \mathbb{F}_p$$

is an isomorphism and other entries in the spectral sequence are all trivial. Similar, we define $V_{(m,n)}$ to be a first quadrant spectral sequence over $\mathbb{F}_p$ such that $E_r^{m,n-m} \cong \mathbb{F}_p$ for $r \geq 2$ and other entries in the spectral sequence are all trivial. We call a first quadrant spectral sequence a universal spectral sequence if it is isomorphic to $U_{(m,n,r)}$ or $V_{(m,n)}$. See Figure 1.

Example 1.7. In this language, Cohen's original calculation, see Proposition 5.5, for the case $k = p$ and $n = 2$ can be expressed as an isomorphism between the Cartan–Leray spectral sequence and

$$V_{(0,0)} \oplus V_{(0,1)} \oplus \bigoplus_{t \geq 1, \varepsilon = 0,1} U_{\big(2t(p-1)-\varepsilon-p, 2t(p-1)-\varepsilon-1, p\big)}.$$

Roughly speaking, the spectral sequence is isomorphic to the sum of two spectral sequences. One spectral sequence is induced by the previous cases where $k' < k$, and another one captures new behavior.

Theorem 1.8. *The spectral sequence for the braid group $B_{2p^2}(\mathbb{R}^2)$ is isomorphic to*

$$\begin{aligned} & V_{\big(p^2-2p, 2p^2-2p\big)} \oplus V_{\big(p^2-2p+1, 2p^2-2p+1\big)} \oplus V_{\big(p^2-2, 2p^2-2\big)} \oplus V_{\big(p^2-1, 2p^2-1\big)} \\ \oplus & \bigoplus_{\substack{t_1 \geq t_2 + 2 - \varepsilon_2 \\ t_2 \geq 1; \varepsilon_1, \varepsilon_2 = 0,1}} U_{\widehat{J}(t_1, \varepsilon_1, t_2, \varepsilon_2)} \oplus U' \end{aligned}$$

where $\widehat{J}(t_1, \varepsilon_1, t_2, \varepsilon_2) = (a, a + 2p^2 - 1, p)$ and

$$a := p\big((2t_2+1)(p-1) + (1-2\varepsilon_2)p - 1\big) - p + (2t_1 + \varepsilon_2)(p-1) - \varepsilon_1,$$

and U' is a spectral sequence whose E_∞ page coincides with

$$V_{(0,0)} \oplus V_{(0,1)} \oplus \bigoplus_{l=1,2,\dots,p-1} V_{\big((p-2)l,(2p-2)l\big)} \oplus V_{\big((p-2)l+1,(2p-2)l+1\big)} \oplus V_{\big((p-2)l,(2p-2)l+1\big)} \oplus V_{\big((p-2)l+1,(2p-2)l+2\big)}.$$

Remark 1.9. See Theorem 5.23 for the general statement, which proves the Arone–Mahowald theorem and Kjaer's theorem up to $2p^2$.

Corollary 1.10. *Let $l = 1, 2, \dots, p-1$ and $\varepsilon = 0, 1$, then*

$$H^*(B_{2p^2}(\mathbb{R}^2); \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p & \text{if } * = 0, 1, (2p-2)l, (2p-2)l+2 \\ \mathbb{F}_p \oplus \mathbb{F}_p & \text{if } * = (2p-2)l + 1 \\ \mathbb{F}_p & \text{if } * = 2p^2 - 2p + \varepsilon, 2p^2 - 2 + \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

Corollary 1.11. *Through degree $2p^2$, A_2 is a free divided power algebra on generators of bidegree*

$$|x_i| = (2p^i - 2, 2p^i), |y_i| = (2p^i - 1, 2p^i)$$

for $i = 1, 2$, where the first degree is the cohomological degree and the second degree is the number of points in configuration spaces.

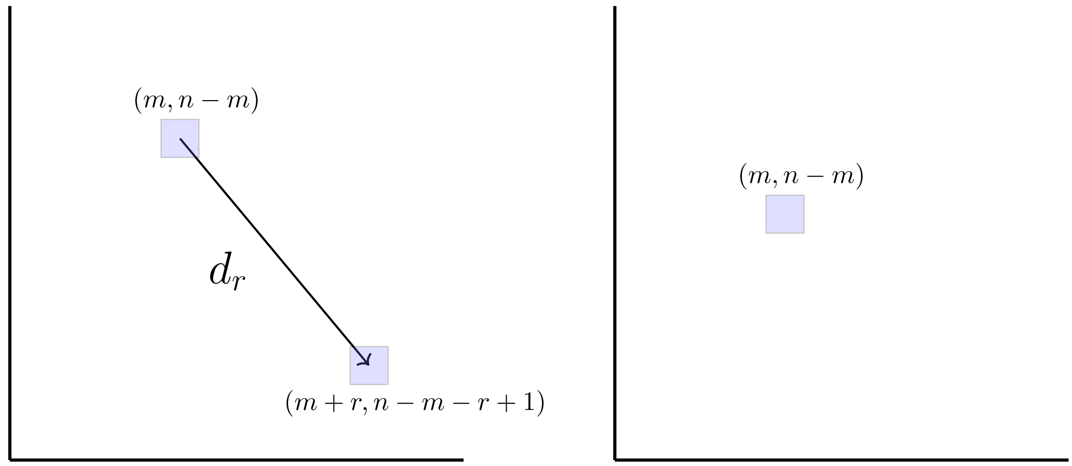

FIGURE 1. $U_{(m,n,r)}$ on the left and $V_{(m,n)}$ on the right.

1.1. Some definitions.

Definition 1.12. Let C be a (co)chain complex with differential d. We call a subcomplex A independent if $dx \in A$ if and only if $dx = dy$ for some $y \in A$.

An immediate result is that the subcomplex A is independent if and only if $H_*(A) \to H_*(C)$ is injective.

Definition 1.13. Let $\{E_r\}$ be a spectral sequence and A_2 a subspace of E_2. We define what it means for A_2 to be independent through E_r recursively as follows:

(1) First, we say that A_2 is independent through E_2 if A_2 is an independent subcomplex of E_2. In this case, we define $A_3 := H_*(A_2)$, thought of as a subspace of E_3.
(2) Assume that independent through E_{r-1} and A_r have been defined. We say that A_2 is independent through E_r if A_r is an independent subcomplex of E_r.

We say that A_2 spans a spectral subsequence if A_2 is independent through E_r for every r. In this case, $\{A_r\}$ forms a spectral sequence with a canonical map of spectral sequences to $\{E_r\}$ that is injective on every page.

Definition 1.14. Let U be a sum of universal spectral sequences. Let μ_i be the distinct partitions of k for $i = 1, 2, \dots, m$, which correspond to the summands $H^*(\Sigma_k, M_{\mu_i})$ in the spectral sequence of $B_k(\mathbb{R}^2)$. We say $\{\mu_1, \mu_2, \dots, \mu_m\}$ spans a spectral subsequence if $\bigoplus_{1 \le i \le m} H^*(\Sigma_k, M_{\mu_i})$ spans a spectral subsequence. We say U provides a model for $\mu_1, \mu_2, \dots, \mu_m$ if

(1) $\{\mu_1, \mu_2, \dots, \mu_m\}$ spans a spectral subsequence, and
(2) There is an injective map of spectral sequences from U to the spectral sequence of $B_k(\mathbb{R}^2)$ with the image $\bigoplus_{1 \le i \le m} H^*(\Sigma_k, M_{\mu_i})$ at the E_2 page.

2. BACKGROUND ON GROUP (CO)HOMOLOGY

This section, together with the next section, are the fundamental tools to solve the problem. The big framework is to apply Cartan-Leray spectral sequence to study the braid group. Hence, we shall review some of the facts about spectral sequences and group cohomology at this section.

2.1. **Basic group (co)homology facts.** Let us recall some group (co)homology facts here which shall be used constantly in this paper. Proofs of the statements can be found at Chapter III of [Bro82].

Proposition 2.1 (Shapiro's lemma). *If $H \leq G$ and M is an H-module, then*

$$H_*(H, M) \cong H_*(G, \operatorname{Ind}_H^G M)$$

and

$$H^*(H, M) \cong H^*(G, \operatorname{Coind}_H^G M).$$

Proposition 2.2 ([Bro82, Prop 3.5.9]). *If the index of a subgroup H in G is finite, then $\operatorname{Ind}_H^G M \cong \operatorname{Coind}_H^G M$.*

Proposition 2.3 ([Bro82, Prop 3.6.1]). *Let G be a group and M a G-module.*

(1) There is a natural isomorphism $H^0(G, M) = M^G$.
(2) For any exact sequence $0 \to M' \to M \to M'' \to 0$ of G-modules and any integer n, there is a natural map $\delta : H^n(G, M'') \to H^{n+1}(G, M')$ such that the sequence

$$0 \to H^0(G, M') \to H^0(G, M) \to H^0(G, M'') \xrightarrow{\delta} H^1(G, M') \to H^1(G, M) \to \dots$$

(3) If Q is an injective $\mathbb{F}_pG$-module, then $H^n(G, Q) = 0$ for $n > 0$.

Proposition 2.4 ([Bro82, Prop 3.9.5]). *Let H, K be subgroups of G and M be a G-module. Denote $\operatorname{res}_H^G : H^*(G, M) \to H^*(H, M)$ as the restriction map and $\operatorname{cor}_H^G : H^*(H, M) \to H^*(G, M)$ as the corestriction map.*

(1) Given $(G : H) < \infty$, $\operatorname{cor}_H^G \circ \operatorname{res}_H^G = [G : H] \cdot id$.
(2) Given $(G : H) < \infty$, $\operatorname{res}_K^G \circ \operatorname{cor}_H^G = \sum_{g \in E} \operatorname{cor}_{K \cap gHg^{-1}}^K \circ \operatorname{res}_{K \cap gHg^{-1}}^{gHg^{-1}} \circ (c(g)^*)^{-1}$, where $c(g) : (H, M) \to (gHg^{-1}, M)$ sends (h, m) to (ghg^{-1}, gm).

Corollary 2.5 ([Bro82, Prop 3.10.2]). *If $|G|$ is invertible in M, then*

$$H^n(G, M) = 0$$

for all $n > 0$.

Corollary 2.6. *If a subgroup H of G is invertible in M and $[G : H] < \infty$, then*

$$H^n(G, \operatorname{Ind}_H^G M) = 0$$

for all $n > 0$.

Proof. This is by 2.5 and Shapiro's lemma. □

2.2. **Spectral sequences and naturality.** The Cartan-Leray spectral sequence theorem is the most fundamental theorem used in this paper. Besides, naturality plays a key role for computing cohomology of B_k where $k > p$.

Theorem 2.7 (The cohomological version of [Bro82, p. 169]). *Let G be a group, C^* be a co-chain complex equipped with a G-action, and let W_* be a free resolution over G, then there is a spectral sequence of the form*

$$E_2^{s,t} \cong H^s(G, H^t(C^*)) \Rightarrow H^{s+t}(\operatorname{Hom}_G(W_*, C^*)),$$

which is natural for equivariant cochain maps.

Corollary 2.8 ([Bro82, p. 173])**.** *If X is a G-space for a group G and M is a G-module, then there is a spectral sequence of the form*

$$E_2^{s,t} = H^s(G, H^t(X, M)) \Rightarrow H^{s+t}(EG \times_G X, M)$$

where the E_2-term here involves the diagonal action of G on $H^*(X, M)$ induced by the acton of G on X and M and $H^*(X/G, M)$ needs to interpreted as cohomology groups with local coefficient if G acts non-trivially on M.

Corollary 2.9 (Cartan–Leray)**.** *If X is a free G-space for a group G and M is a G-module, then there is a spectral sequence of the form*

$$E_2^{s,t} = H^s(G, H^t(X, M)) \Rightarrow H^{s+t}(X/G, M).$$

Corollary 2.10 ([Bro82, p. 175])**.** *Let X be a G-space. If N is a normal subgroup of G which acts freely on X, then*

$$H^*(EG \times_G X) \cong H^*(G/N \times_{G/N} X/N)$$

with any G/N-module of coefficients.

For our purpose, we can study fibration of the form

$$\mathrm{Conf}_k \to B_k \to B\Sigma_k.$$

Corollary 2.11. *Given X be a G-space and Y be a H-space such that $H \leq G$. Let $f : X \to Y$ be a continuous H-equivariant map, then the induced map between spectral sequences*

$$f^* : H^*(H, H^*(Y)) \to H^*(G, H^*(X))$$

commutes with the differentials at every page.

Proof. By assumption, $f : X \to Y$ induces a map at the co-chain level $f : C^*(Y) \to C^*(X)$. This map is a H-homomorphism. By the universal mapping property, there is an induced G-homomorphism

$$f^* : \mathrm{Ind}_H^G C^*(Y) \to C^*(X).$$

This induces a map between spectral sequences by 2.7 and

$$H^*(G, \mathrm{Ind}_H^G H^*(Y)) \cong H^*(H, H^*(Y))$$

by Shapiro's lemma. □

Proposition 2.12 ([Bro82, p. 169])**.** *If $\tau : C \to C'$ is a quasi-isomorphism of two G-chain complexes, then τ induces an isomorphism $H^*(\mathrm{Hom}_G(W_*, C)) \xrightarrow{\cong} H^*(\mathrm{Hom}_G(W_*, C'))$, where W_* is a free resolution over G.*

In order to study spectral sequence of B_k for large k, we 'break' it into smaller pieces by studying spectral sequence of product of B_k for small k. Hence, we need to study the spectral sequence regarding to the product and the wreath product. For that, we introduce the Lyndon-Hochschild-Serre spectral sequence here.

Theorem 2.13 (Lyndon–Hochschild–Serre)**.** *For any group extension*

$$1 \to H \to G \to Q \to 1$$

and any G-module M, there is a spectral sequence

$$E_2^{s,t} \cong H^s(Q, H^t(H, M)) \Rightarrow H^{s+t}(G, M).$$

We need Lyndon–Hochschild–Serre spectral sequence to study the group extension regarding to the wreath product of symmetric groups. With the $\mathbb{F}_p$-coefficient, [Nak61, Thm 3.3] has proven that

$$H^*(\Sigma_l \wr \Sigma_m, \mathbb{F}_p) \cong H^*(\Sigma_l, H^*(\Sigma_m, \mathbb{F}_p)^{\otimes l}).$$

We can expand this idea by considering a more general coefficient.

Theorem 2.14 ([Lea07, Thm 2.1])**.** *Let X be a connected space of finite type with fundamental group G, and let M be an $\mathbb{F}_p$-free G-module, then the spectral sequence with coefficients in $M^{\otimes k}$ (over $\mathbb{F}_p$) for the fibration*

$$(X)^k \to (X)^k \times_{\Sigma_k} E\Sigma_k \to B\Sigma_k$$

collapses at the E^2-page.

Note that $(X)^k \times_{\Sigma_k} E\Sigma_k$ is the classifying space of $\Sigma_k \wr G$.

2.3. **Cohomology of symmetric groups.** We will run into a lot of scenarios where cohomology of symmetric groups appear in our Cartan-Leray spectral sequences. Hence, we need a tool to study them.

Recall a classical theorem by May.

Theorem 2.15 ([May70, p. 123])**.** *Consider group cohomology $H^*(\Sigma_p; M)$ and let β denotes the mod p Bockstein, then*

(1) for $M = \mathbb{F}_p$, $H^*(\Sigma_p; \mathbb{F}_p) = \wedge_{\mathbb{F}_p}[v] \otimes \mathbb{F}_p[\beta v]$ as an algebra where v is a class of degree $2(p-1)-1$; and

(2) for $M = \mathbb{F}_p^{sgn}$, $H^s(\Sigma_p; \mathbb{F}_p^{sgn})$ has the additive basis $\{(\beta v)^s \beta^\varepsilon v'\}$ where v' is a class of degree $p-2$, $\varepsilon = 0, 1$ and $s \geq 0$.

This tells us cohomology of Σ_p over $\mathbb{F}_p$. However, for our purpose, we need to study cohomology of Σ_k over $\mathbb{F}_p$ for an arbitrary positive integer k.

Dyer-Lashof operation provides an extreme helpful way to understand (co)homology of symmetric groups with coefficient either trivial or sign representation, even though it has boarder applications including general linear groups. To understand the full potential of Dyer-Lashof operation, readers can access Bernard's paper here [Ber23], where (co)homology of symmetric group is just one of its applications.

Consider two types of homology operations:

- the untwisted Dyer-Lashof operations
$$Q^s : H_n(\Sigma_g, \mathbb{F}_p) \to H_{n+2s(p-1)}(\Sigma_{pg}, \mathbb{F}_p)$$
$$\beta Q^s : H_n(\Sigma_g, \mathbb{F}_p) \to H_{n+2s(p-1)-1}(\Sigma_{pg}, \mathbb{F}_p)$$
for $n \in \mathbb{Z}$ and $s \in \mathbb{Z}$.
- the twisted Dyer-Lashof operations
$$Q^{s+\frac{1}{2}} : H_n(\Sigma_g, \mathbb{F}_p^{sgn}) \to H_{n+(2s+1)(p-1)}(\Sigma_{pg}, \mathbb{F}_p^{sgn})$$
$$\beta Q^{s+\frac{1}{2}} : H_n(\Sigma_g, \mathbb{F}_p^{sgn}) \to H_{n+(2s+1)(p-1)-1}(\Sigma_{pg}, \mathbb{F}_p^{sgn})$$
for $n \in \mathbb{Z}$ and $s \in \mathbb{Z}$.

They satisfy a list of properties:

Proposition 2.16 ([Ber23, Thm. 6.21])**.** *Let $M = \mathbb{F}_p$ or $\mathbb{F}_p^{sgn}$. The Dyer-Lashof operations consist of natural transfomations*

$$Q^s : H_n(\Sigma_g, M) \to H_{n+2s(p-1)}(\Sigma_{pg}, M)$$
$$\beta Q^s : H_n(\Sigma_g, M) \to H_{n+2s(p-1)-1}(\Sigma_{pg}, M)$$

for every $s \in \mathbb{Z} \sqcup \mathbb{Z} + \frac{1}{2}$ satisfying the following relations.

(1) The composition of trivial coefficient and sign representation is always trivial, i.e.,

$$Q^{s'} Q^{s+\frac{1}{2}} = Q^{s'+\frac{1}{2}} Q^s = 0$$

for any s and s'.

(2) $Q^s = 0$ if $2s < n$.

(3) $\beta Q^s = 0$ if $2s \leq n$.

(4) If $x \in H_n(\Sigma_g)$ with $n = 2s$, then $Q^s(x) = x^p$.

By digging into the original definition of Dyer-Lashof operations, not only every element in $H_*(\Sigma_{p^2})$ comes from $H_*(\Sigma_p \wr \Sigma_p)$, but also we know which summand of $H_*(\Sigma_p \wr \Sigma_p)$ specifically since

$$H_s(\Sigma_p \wr \Sigma_p) \cong \bigoplus_{s_1+s_2=s} H_{s_1}(\Sigma_p, H_{s_2}(\Sigma_p)).$$

Note that we always have the following surjective 'corestriction' map,

$$H_*(\Sigma_p \wr \Sigma_p) \to H_*(\Sigma_{p^2}).$$

Consider tuples

$$I = (\varepsilon_1, s_1, \varepsilon_2, s_2, \ldots, \varepsilon_k, s_k)$$

where ε_i is either 0 or 1, s_i is either in $\mathbb{Z}$ or in $(\mathbb{Z} + \frac{1}{2})$. Any such I determines a word in the Dyer-Lashof operations

$$Q^I := \beta^{\varepsilon_1} Q^{s_1} \ldots \beta^{\varepsilon_k} Q^{s_k}.$$

We say such a sequence I admissible if both of the following conditions hold:

(1) For untwisted operations, $s_i \in \mathbb{Z}$ for all $1 \leq i \leq k$; for twisted operations, $s_i \in (\mathbb{Z} + \frac{1}{2})$ for all $1 \leq i \leq k$.

(2) $ps_i - \varepsilon_i \geq s_{i-1}$ for all $1 < i \leq k$.

We call $l(I) = k$, the length of I, and $e(I) := 2s_1 - \varepsilon_1 - \sum_{i=2}^{k}(2s_i(p-1) - \varepsilon_i)$, the excess of I.

Theorem 2.17 ([Ber23, Thm. 7.13])**.** *Let $M = \mathbb{F}_p$ or $\mathbb{F}_p^{sgn}$, then $H_*(\Sigma_g, M)$ is isomorphic to the free commutative algebra on $\{Q^I x\}$ as a $\mathbb{F}_p$-vector space, where $x \in H_0(\Sigma_1, M)$, $Q^I x \in H_t(\Sigma_{p^{l(I)}}, M)$ for*

$$t = \sum_{i=1}^{k} (2s_i(p-1) - \varepsilon_i), \quad \sum_I p^{l(I)} = g,$$

and Q^I rangers over all admissible compositions of Dyer-Lashof operations subject to $e(I) + \varepsilon_1 > 0$.

Remark 2.18. Let us call the free commutative algebra above S, we give a default total ordering to elements in S. For element $s_1 s_2 \ldots s_j$ where $s_1 \leq s_2 \leq \cdots \leq s_j$, we can only have $s_i = s_{i+1}$ if homology degree of s_i is odd with sign representation and even with trivial coefficient.

Example 2.19. Let $M = \mathbb{F}_p$ or $\mathbb{F}_p^{sgn}$. For $x \in H_0(\Sigma_1, M)$, we have

$$Q^1(x) \in H_{2(p-1)}(\Sigma_p, \mathbb{F}_p), \beta Q^1(x) \in H_{2(p-1)-1}(\Sigma_p, \mathbb{F}_p)$$
$$Q^{\frac{1}{2}}(x) \in H_{(p-1)}(\Sigma_p, \mathbb{F}_p^{sgn}), \beta Q^{\frac{1}{2}}(x) \in H_{p-2}(\Sigma_p, \mathbb{F}_p^{sgn})$$

This is related to 2.15. Hence, $v = (\beta Q^1(x))^*, \beta v = (Q^1(x))^*$, and $v' = (\beta Q^{\frac{1}{2}}(x))^*, \beta v' = (Q^{\frac{1}{2}}(x))^*$.

Example 2.20. For $x \in H_0(\Sigma_1, \mathbb{F}_p)$ and $l \in \mathbb{N}$, we have

$$H_*(\Sigma_{lp}, \mathbb{F}_p) \cong H_*(\Sigma_{lp+1}, \mathbb{F}_p) \cong \ldots \cong H_*(\Sigma_{lp+(p-1)}, \mathbb{F}_p).$$

Example 2.21. For $x \in H_0(\Sigma_1, \mathbb{F}_p^{sgn})$ and $l \in \mathbb{N}$, we have

$$H_*(\Sigma_{lp}, \mathbb{F}_p^{sgn}) \cong H_*(\Sigma_{lp+1}, \mathbb{F}_p^{sgn}),$$
$$H_*(\Sigma_{lp+2}, \mathbb{F}_p^{sgn}) = H_*(\Sigma_{lp+3}, \mathbb{F}_p^{sgn}) = \cdots = H_*(\Sigma_{lp+(p-1)}, \mathbb{F}_p^{sgn}) = 0.$$

This is because $x^2 \in H_0(\Sigma_2, \mathbb{F}_p^{sgn}) = 0$.

Corollary 2.22. *For $H_*(\Sigma_{p^r}, \mathbb{F}_p^{sgn})$ with $r \geq 1$, the first non-trivial admissible set I of length r satisfying the conditions on theorem 2.17 has the form*

$$I = (\varepsilon_1, \frac{p^{r-1}}{2}, 0, \frac{p^{r-2}}{2}, 0, \frac{p^{r-3}}{2}, \ldots, 0, \frac{p}{2}, 0, \frac{1}{2}),$$

and for such I, $* = p^r - 1 - \varepsilon_1$.

Proof. For $I = (\varepsilon_1, s_1, \varepsilon_2, s_2, \ldots, \varepsilon_r, s_r)$, this is subject to

$$ps_r - \varepsilon_r \geq s_{r-1}$$
$$ps_{r-1} - \varepsilon_{r-1} \geq s_{r-2}$$
$$\ldots\ldots$$
$$ps_2 - \varepsilon_2 \geq s_1$$

and

$$2s_r > 0$$
$$2s_{r-1} > 2s_r(p-1) - \varepsilon_r$$
$$\ldots\ldots$$
$$2s_1 > \sum_{i=2}^{r} 2s_i(p-1) - \varepsilon_i$$

For the first admissible set I, we start with $s_r = \frac{1}{2}$, then

$$\frac{p-1-\varepsilon_r}{2} < s_{r-1} \leq \frac{p-\varepsilon_r}{2}.$$

If $\varepsilon_r = 1$, then s_{r-1} does not exist.

If $\varepsilon_r = 0$, then $s_{r-1} = \frac{p}{2}$. Then we continue this process to get

$$I = (\varepsilon_1, \frac{p^{r-1}}{2}, 0, \frac{p^{r-2}}{2}, 0, \frac{p^{r-3}}{2}, \ldots, 0, \frac{p}{2}, 0, \frac{1}{2}).$$

Note that this is related to

$$\beta^{\varepsilon_1} Q^{\frac{p^{r-1}}{2}} \ldots Q^{\frac{p}{2}} Q^{\frac{1}{2}}(x) \in H_{p^r-1-\varepsilon_1}(\Sigma_{p^r}, \mathbb{F}_p^{sgn}).$$

for $x \in H_0(\Sigma_1, \mathbb{F}_p^{sgn})$. □

Let us take a close look at the co-restriction map we mentioned before, and we shall make it into a general case.

Lemma 2.23. *Let $\Sigma_p \wr \Sigma_{p^{r-1}}$ be a subgroup of Σ_{p^r} and $\varepsilon = 0$ or 1, then we have the surjective transfer map with coefficient of sign representation*

$$f : H^{p^r-1-\varepsilon}(\Sigma_p \wr \Sigma_{p^{r-1}}, \mathbb{F}_p^{sgn}) \to H^{p^r-1-\varepsilon}(\Sigma_{p^r}, \mathbb{F}_p^{sgn})$$

with the image isomorphic to

$$H^0(\Sigma_p, H^{p^r-1-\varepsilon}((\Sigma_{p^r-1})^p, (\mathbb{F}_p^{sgn})^{\otimes p}) \oplus H^{p-1-\varepsilon}(\Sigma_p, (H^{p^{r-1}-1}(\Sigma_{p^{r-1}}, \mathbb{F}_p^{sgn}))^{\otimes p}).$$

as a direct summand of $H^{p^r-1-\varepsilon}(\Sigma_p \wr \Sigma_{p^{r-1}}, \mathbb{F}_p^{sgn})$.

Proof. This is by the original definition of Dyer-Lashof operations.

For $l(I) = r$, subject to the conditions on I, we get that

$$\beta^\varepsilon Q^{\frac{p^{r-1}}{2}} \dots Q^{\frac{p}{2}} Q^{\frac{1}{2}}(x) \in H_{p^r-1-\varepsilon}(\Sigma_{p^r}, \mathbb{F}_p^{sgn}),$$

and this is from $H_{p-1-\varepsilon}(\Sigma_p, (H_{p^{r-1}-1}(\Sigma_{p^{r-1}}, \mathbb{F}_p^{sgn}))^{\otimes p})$.

For $l(I) < r$, the corresponding terms all come from $H_0(\Sigma_p, H_{p^r-1-\varepsilon}((\Sigma_{p^r-1})^p, (\mathbb{F}_p^{sgn})^{\otimes p})$ by the total ordering argument after theorem 2.17. $\square$

Proposition 2.24. *For $M = \mathbb{F}_p$ or $\mathbb{F}_p^{sgn}$ and $1 \le l < p$,*

$$H_*(\Sigma_{lp^r}, M) \cong H_*(\Sigma_l \wr \Sigma_{p^r}, M).$$

Proof. First, we have an injective $\mathbb{F}_p$-vector space homomorphism

$$H_*(\Sigma_{lp^r}, M) \to H_*(\Sigma_l \wr \Sigma_{p^r}, M) \cong (H_*(\Sigma_{p^r}, M)^{\otimes l})_{\Sigma_l}.$$

By 2.17, both sides have the same basis. This is because for admissible I, it has length ranging from $l(I) = 1$ to $l(I) = r$ for both sides. Besides, for element $s_1 s_2 \dots s_j$ in this free commutative algebra given a default total ordering $s_1 \le s_2 \le \dots \le s_j$, we can only have $s_i = s_j$ if homology degree of s_i is n for $(-1)^{n+1} = 1$ with sign representation and $(-1)^n = 1$ with trivial coefficient. This extra condition coincide with Σ_l coinvariant property for $H_*(\Sigma_{p^r}, M)^{\otimes l}$. $\square$

3. Configuration spaces

In the spectral sequence related to $H^*(\Sigma_k, H^*(\mathrm{Conf}_k, \mathbb{F}_p))$, we can simplify each entry by getting rid of many trivial components, which can be accomplished by looking at partitions of k and related submodules of $H^*(\mathrm{Conf}_k)$. However, this is not enough because we lack information about the entries in the top rows and bottom rows of the spectral sequence. For the top rows, we may fix the issue by looking at cohomology dimension of B_k. As for the bottom rows, Dyer-Lashof operation (from last section) provides a solution.

3.1. Maps out, maps in, and partitions.

Definition 3.1. We call the projection map

$$\mathrm{Conf}_k \to \mathrm{Conf}_{k_1} \times \dots \times \mathrm{Conf}_{k_l},$$

a "map out" if $k_1 + \dots + k_l = k$.

We can consider its dual notion.

Definition 3.2. Let $\varphi : \coprod_{i=1,2,\ldots,l} \mathbb{R}^n \to \mathbb{R}^n$ be any orientation preserving embedding, and we call the dashed map of the following form

$$\begin{tikzcd}
\mathrm{Conf}_{k_1}(\mathbb{R}^n) \times \cdots \times \mathrm{Conf}_{k_l}(\mathbb{R}^n) \arrow[d, hook] \arrow[dr, dashed] & \\
\mathrm{Conf}_k(\coprod \mathbb{R}^n) \arrow[r, "\mathrm{Conf}_k(\varphi)"'] & \mathrm{Conf}_k(\mathbb{R}^n)
\end{tikzcd}$$

a "map in" if $k_1 + \cdots + k_l = k$.

Note that the any two choices of φ are the same, up to homotopy.

Fix an integer k, and consider a (unordered) partition λ of k of length r such that

$$k_1 + k_2 + \cdots + k_r = k.$$

This corresponds to a subgroup

$$\Sigma_\lambda \leq \Sigma_k,$$

which is defined to be the product of $\Sigma_a \wr \Sigma_{k_b}$'s where a is the number of times where k_b appears. This also corresponds to a Σ_λ-equivariant map out

$$f : \mathrm{Conf}_k \to \mathrm{Conf}_{k_1} \times \mathrm{Conf}_{k_2} \times \cdots \times \mathrm{Conf}_{k_l} =: \mathrm{Conf}_\lambda .$$

3.2. $H^*(\mathrm{Conf}_k(\mathbb{R}^n))$ **and induced submodules related to partitions of** k**.** Let λ be a partition of k of length r, and consider the following commutative (up to homotopy) diagram

$$\begin{tikzcd}
\mathrm{Conf}_\lambda \arrow[r, "i"] \arrow[dr, "\mathrm{id}"'] & \mathrm{Conf}_k \arrow[d, "p"] \\
& \mathrm{Conf}_\lambda
\end{tikzcd}$$

where i is a map in and p is a map out. This shows that the induced map p^* is injective.

$$\begin{tikzcd}
H^{(k-r)(n-1)}(\mathrm{Conf}_\lambda) & H^{(k-r)(n-1)}(\mathrm{Conf}_k) \arrow[l, "i^*"'] \\
& H^{(k-r)(n-1)}(\mathrm{Conf}_\lambda) \arrow[u, "p^*"'] \arrow[ul, "id"]
\end{tikzcd}$$

Note that p^* is a Σ_λ-module homomorphism and $H^{(k-r)(n-1)}(\mathrm{Conf}_k)$ is a Σ_k-module. Hence, by the universal mapping property, this induces a map

$$\bar{p^*} : \mathrm{Ind}_{\Sigma_\lambda}^{\Sigma_k} H^{(k-r)(n-1)}(\mathrm{Conf}_\lambda) \to H^{(k-r)(n-1)}(\mathrm{Conf}_k).$$

such that the following diagram commutes

$$\begin{tikzcd}
H^{(k-r)(n-1)}(\mathrm{Conf}_\lambda) & H^{(k-r)(n-1)}(\mathrm{Conf}_k) \arrow[l, "i^*"'] & \\
& H^{(k-r)(n-1)}(\mathrm{Conf}_\lambda) \arrow[u, "p^*"'] \arrow[ul, "id"] \arrow[r] & \mathrm{Ind}_{\Sigma_\lambda}^{\Sigma_k} H^{(k-r)(n-1)}(\mathrm{Conf}_\lambda) \arrow[ul, dashed, "\bar{p^*}"']
\end{tikzcd}$$

Note that $\bar{p^*}$ is also injective, and this is by [Sin06, Cor 4.6].

Definition 3.3. Given a partition λ of k of length r, we call

$$M_\lambda := \operatorname{im} \bar{p^*} \cong \operatorname{Ind}_{\Sigma_\lambda}^{\Sigma_k} H^{(k-r)(n-1)}(\mathrm{Conf}_\lambda),$$

an induced submodule related to the partition λ.

Let λ run through all (unordered) partitions of k of length r. By the universal property of the direct sum, we have a Σ_k-modules homomorphism

$$\oplus \bar{p^*} : \bigoplus_\lambda M_\lambda \to H^{(k-r)(n-1)}(\mathrm{Conf}_k).$$

Proposition 3.4 ([Sin06, Thm 2.10])**.** *There is an isomorphism of Σ_k-modules*

$$H^{(k-r)(n-1)}(\mathrm{Conf}_k) \cong \bigoplus_{\lambda;|\lambda|=r} M_\lambda,$$

and hence

$$H^*(\mathrm{Conf}_k) \cong \bigoplus_{r=1}^{k} \bigoplus_{\lambda;|\lambda|=r} M_\lambda.$$

Definition 3.5. Let k be an integer, and consider all the partitions of k. For a partition of the form $(k_1, k_2, \dots, k_r)$ of length r, it corresponds a Σ_k-submodule M_λ of $H^*(\mathrm{Conf}_k(\mathbb{R}^2), \mathbb{F}_p)$. We call a partition cohomologically non-trivial if $H^*(\Sigma_k, M_\lambda) \neq 0$ for $* > 0$.

With the help of cohomology dimension of $B_k(\mathbb{R}^n)$, we will be able to classify all cohomologically non-trivial partitions for a fixed k. See 3.17.

Proposition 3.6 ([Knu18, Prop 3.2.6])**.**

$$M_k := H^{(k-1)(n-1)}(\mathrm{Conf}_k(\mathbb{R}^n))$$

is a free Σ_{k-1}-module as a sub Σ_k-module.

We wish to understand the first column of the spectral sequence. In other words, we need to compute $H^0(\Sigma_k, M_\lambda)$ for a given partition λ.

Remark 3.7. For the case $k = p$, it is called the Invariants Theorem in Cohen's paper. Note that we can easily expand to the general case. For the proof, [Knu18, p. 81] provides a more reader-friendly version here.

Theorem 3.8 (Invariants Theorem)**.** *For any prime $p > 3$,*

$$H^0(\Sigma_k, M_\lambda) = \begin{cases} \mathbb{F}_p & \textit{if } \lambda = (1,1,\dots,1) \\ (M_2)^{\Sigma_2} & \textit{if } \lambda = (2,1,\dots,1) \\ 0 & \textit{otherwise} \end{cases}$$

3.3. **Cohomology dimension of $B_k(\mathbb{R}^n)$.** We denote by $cohdim_{\pm\mathbb{Z}}(M)$ the smallest integer with the property that

$$H^i(M, \pm\mathbb{Z}) = 0, \forall i > cohdim_{\pm\mathbb{Z}}(M).$$

Theorem 3.9 ([Kal08, Thm 1.1])**.** *Let M be a compact manifold of dimension $n \geq 1$, with boundary ∂M, and let $U \subset M$ be a closed subset such that $U \cap \partial M = \emptyset$ and $M - U$ connected. We denote by r the connectivity of the quotient $M/(U \cup \partial M)$ if $U \cup \partial M \neq \emptyset$. We assume $0 \leq r < \infty$ and $k \geq 2$. Then*

$$cohdim_{\pm\mathbb{Z}}(B_k(M - U)) \leq (n-1)k - r$$

When M is even dimensional orientable, then replace $cohdim_{\pm\mathbb{Z}}$ by cohdim.

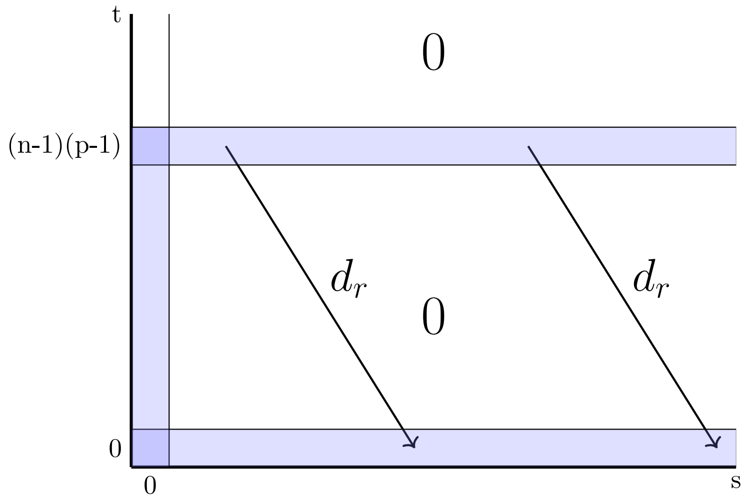

FIGURE 2. E_2-page of $\mathrm{Conf}_p(\mathbb{R}^n) \to B_p(\mathbb{R}^n) \to B\Sigma_p$ where $r = (n-1)(p-1)+1$.

Corollary 3.10. *$H^i(B_k(\mathbb{R}^n), \mathbb{F}_p) = 0$ for any $i \geq (n-1)k - n + 2 = (n-1)(k-1)+1$.*

Proof. Pick $M = S^n$ and $U = \{pt\}$ where n is even, then $S^n - pt \cong \mathbb{R}^n$ and $r = n-1$ in this case, then we use the universal coefficient theorem to pass from $\mathbb{Z}$ to $\mathbb{F}_p$. The statement follows since $B_k(\mathbb{R}^n)$ is orientable. □

Note that Arnold [Arn14, p. 32] proved the case when $n = 2$.

3.4. **$k = p$ case from Cohen.** In this section, we revisit Cohen's calculation for the case $k = p$ using the tools we have seen so far. The original calculation can be found at [FM76, p. 207]. We arrange this section here because this lays the foundation for later sections.

The main strategy is to apply Cartan-Leray spectral sequence to fiber bundle

$$\mathrm{Conf}_p(\mathbb{R}^n) \to B_p(\mathbb{R}^n) \to B\Sigma_p.$$

This produces a spectral sequence with

$$E_2^{s,t} \cong H^s(\Sigma_p, H^t(\mathrm{Conf}_p(\mathbb{R}^n), \mathbb{F}_p)) \Rightarrow H^{s+t}(B_p(\mathbb{R}^n), \mathbb{F}_p).$$

By 3.4 and 2.1, we can determine the non-invariant part of E_2 page of the spectral sequence. It turns out that there are only two cohomologically non-trivial partitions of p, and they are (p) and $(1, 1, \ldots, 1)$. They are corresponding to the top row and the bottom row in the spectral sequence. As for the invariant part, this can be determined by 3.8. See figure 2.

By 3.10, we know that the differentials at page $(p-1)(n-1)+1$ between these two rows above must be isomorphisms. Hence, the E_∞ page looks like a chop-off form. See figure 3.

Remark 3.11. As for the terminology in Cohen's paper, if we combine 3.4 and 2.1, it is called Vanishing Theorem. Another key statement is called Periodicity Theorem. However, 3.10 is a 'stronger' version than the Periodicity Theorem since it can used for the general case. See [Knu18, p. 81] for the statements and the proofs. See [Knu18, p. 112] for why Periodicity Theorem fails for the general case.

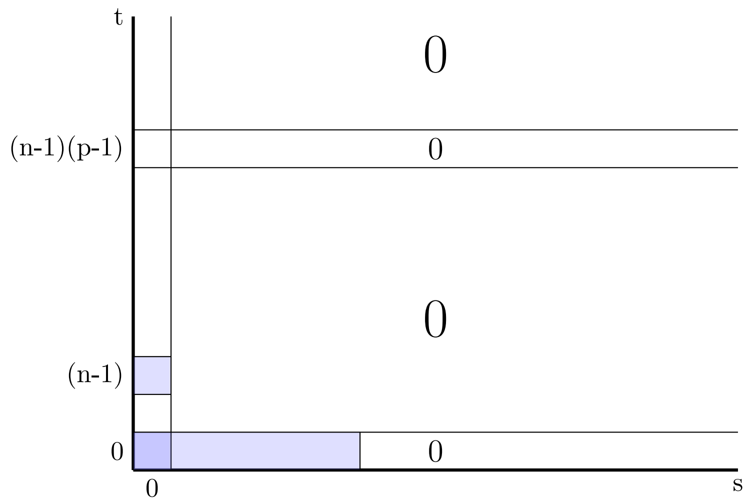

FIGURE 3. E_∞-page of $\mathrm{Conf}_p(\mathbb{R}^n) \to B_p(\mathbb{R}^n) \to B\Sigma_p$.

Theorem 3.12 (Cohen)**.** *There is an isomorphism*

$$H^*(B_p(\mathbb{R}^n), \mathbb{F}_p) \cong I \times_{\mathbb{F}_p} \frac{H^*(\Sigma_p, \mathbb{F}_p)}{H^{>(n-1)(p-1)}(\Sigma_p, \mathbb{F}_p)}$$

where

$$I := H^*(\mathrm{Conf}_p(\mathbb{R}^n), \mathbb{F}_p)^{\Sigma_p} = \begin{cases} \wedge_{\mathbb{F}_p}(\alpha_{n-1}) & n \text{ is even} \\ \mathbb{F}_p & n \text{ is odd} \end{cases}$$

where α_{n-1} is a non-zero class of degree $n-1$ in $H^*(\mathrm{Conf}_p(\mathbb{R}^n), \mathbb{F}_p)$ and p is any prime greater than 3*.*

Corollary 3.13. *If $n = 2$ and $p > 3$, then*

$$H^s(B_p(\mathbb{R}^2), \mathbb{F}_p) = \begin{cases} \mathbb{F}_p & s = 0 \\ \mathbb{F}_p & s = 1 \\ 0 & \text{otherwise} \end{cases}$$

where for $s = 0$, $\mathbb{F}_p$ comes from $H^0(\Sigma_p, \mathbb{F}_p)$, and for $s = 1$, $\mathbb{F}_p$ comes from $H^1(\mathrm{Conf}_p(\mathbb{R}^2), \mathbb{F}_p)^{\Sigma_p}$.

3.5. **The Arone–Mahoward Theorem.** The following statement, originally proved by [AM99, Thm 4.4], is important because most of top rows in our spectral sequence are actually trivial.

Theorem 3.14 (Arone–Mahowald)**.**

$$H^*(\Sigma_k, H^{(k-1)}(\mathrm{Conf}_k(\mathbb{R}^2)) =: H^*(\Sigma_k, M_k) \cong 0$$

unless k is of the form p^r or $2p^r$ for $r \in \mathbb{N}$.

Remark 3.15. The previous statement also holds over $\mathbb{R}^n$, where n is general. Potentially, this could also be proven using spectral sequence argument and 3.10.

To determine the second page of the spectral sequence, we need to compute $H^*(\mathrm{Conf}_k, \mathbb{F}_p)$. For that, it is equivalent to find all partitions of k.

Definition 3.16. Given two partitions λ and λ' of an integer k. We say λ' is p-subordinate to λ if λ' is obtained from λ by replacing pi with $(i, i, \dots, i)$ for p many i's. We say λ is a top partition if λ is not p-subordinate to μ for any partition μ of the integer k.

Remark 3.17. Note that for the all partitions of k, we can get rid of many cohomologically trivial partitions (See Definition 3.5) since its cohomology is trivial after applying $H^*(\Sigma_k, -)$ by 2.5. Hence, we should only keep the partitions which only have 1, 2, p, $2p$, p^2, $2p^2$, and so on (repeating on p^r and $2p^r$). This is also by 3.14. Besides, for the partitions with 2 involved, we should only keep these partitions with 2 appearing 1 time, p times, $p+1$ times, $2p$ times, $2p+1$ times, and so on (repeating on $l \cdot p$ or $l \cdot p + 1$)). This is because for l, the number of 2's in the partition, we have by 2.21

$$H^*(\Sigma_l \wr \Sigma_2, M_2^{\otimes l}) \cong H^*(\Sigma_l, \mathbb{F}_p[1]^{\otimes l}) \cong H^*(\Sigma_l, \mathbb{F}_p^{sgn}) = 0$$

if l is not of the form mentioned above.

Hence, we have simplified the E_2 page of the spectral sequence from all partitions of k into certain patterns of partitions by 3.17. Next, we will show that there are differentials connecting partitions of form

$$(p^r, p^r, \dots, p^r, \dots)$$

for p many p^r with partitions of the form

$$(p^{r+1}, \dots).$$

3.6. **Maps between spectral sequence.** In this section, we shall rely on 2.11 to study the maps between two spectral sequences induced by a Σ_λ-equivalent map out $f : \mathrm{Conf}_k \to \mathrm{Conf}_\lambda$, where λ is any partition of k of length r. This induces a submodule $M_\lambda \subseteq H^{k-r}(\mathrm{Conf}_k)$ such that the following diagram commutes

$$\begin{tikzcd}
H^{k-r}(\mathrm{Conf}_\lambda) \arrow[r] \arrow[d, hook] & M_\lambda \subseteq H^{k-r}(\mathrm{Conf}_k) \\
\mathrm{Ind}_{\Sigma_\lambda}^{\Sigma_k} H^{k-r}(\mathrm{Conf}_\lambda) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

Lemma 3.18. *The induced map between spectral sequences at $E_2^{(s,k-r)}$*

$$f^* : H^s(\Sigma_k, \mathrm{Ind}_{\Sigma_\lambda}^{\Sigma_k} H^{k-r}(\mathrm{Conf}_\lambda)) \to H^s(\Sigma_k, H^{k-r}(\mathrm{Conf}_k)) \cong H^s(\Sigma_k, \bigoplus_{\lambda; |\lambda| = r} M_\lambda)$$

is an embedding onto $H^s(\Sigma_k, M_\lambda)$.

Proof. This is by the functoriality of the spectral sequence, and the isomorphism at the target is by 3.4. $\square$

Furthermore, Let $G = \Sigma_k$ and $K = \Sigma_\lambda$, then $f : \mathrm{Conf}_k \to \mathrm{Conf}_\lambda$ is also H-equivalent for the fixed partition λ of k, where $H \leqslant K$.

Lemma 3.19. *Let $M = H^{k-r}(\mathrm{Conf}_\lambda)$. The induced map between spectral sequences at $E_2^{(s,k-r)}$*

$$f^* : H^s(G, \mathrm{Ind}_H^G M) \to H^s(G, \mathrm{Ind}_K^G M) \subseteq H^s(G, H^{k-r}(\mathrm{Conf}_k))$$

coincides with the corestriction map $\mathrm{cor}_H^K : H^s(H, M) \to H^s(K, M)$.

Proof. $H^s(G, \operatorname{Ind}_K^G M)$ as a summand in $H^s(G, H^{k-r}(\mathrm{Conf}_k))$ is by Lemma 3.18. Let $F_\bullet$ be a free resolution over G. Then the following diagram is commutative:

$$\begin{tikzcd} F_\bullet \otimes_G \mathbb{F}G \otimes_H M \arrow[r, "\pi"] & F_\bullet \otimes_G \mathbb{F}G \otimes_K M \\ F_\bullet \otimes_H M \arrow[r, "\pi"'] \arrow[u, "\cong"] & F_\bullet \otimes_K M \arrow[u, "\cong"'] \end{tikzcd}$$

Applying $H^*(G, -)$, we have

$$\begin{tikzcd} H^*(G, \operatorname{Ind}_H^G M) \arrow[r] & H^*(G, \operatorname{Ind}_K^G M) \\ H^*(H, M) \arrow[r, "\mathrm{cor}_H^K"'] \arrow[u, "\cong"] & H^*(K, M) \arrow[u, "\cong"'] \end{tikzcd}$$

where two vertical maps are isomorphism due to Shapiro's lemma. □

Similarly, let $g : \mathrm{Conf}_\lambda \to \mathrm{Conf}_k$ be a map in, which is H-equivalent. By universal mapping property again, we have

$$\begin{tikzcd} & \operatorname{Hom}_H(\mathbb{F}G, H^{k-r}(\mathrm{Conf}_\lambda)) \arrow[d] \\ H^{k-r}(\mathrm{Conf}_k) \arrow[r] \arrow[ur, dashed] & H^{k-r}(\mathrm{Conf}_\lambda) \end{tikzcd}$$

Lemma 3.20. *Let $M = H^{k-r}(\mathrm{Conf}_\lambda)$. The induced map between spectral sequences at $E_2^{(s,k-r)}$*

$$g^* : H^s(G, \operatorname{Ind}_K^G M) \subseteq H^s(G, H^{k-r}(\mathrm{Conf}_k)) \to H^s(G, \operatorname{Ind}_H^G M)$$

coincides with the restriction map $\mathrm{res}_H^K : H^s(K, M) \to H^s(H, M)$.

Proof. It is similar to Lemma 3.19. □

4. Universal spectral sequence

In this section, we will first construct a "universal" filtered cochain complex D such that the spectral sequence for the complex is isomorphic to the $U_{(m,n,r)}$ or the $V_{(m,n)}$ from Definition 1.6. By studying the spectral sequence for the filtered complex

$$\operatorname{Hom}_{\Sigma_p}(W_*, (D)^{\otimes p})$$

and the filtered map

$$D \to \operatorname{Hom}_{\Sigma_k}(F_*, C^*(\mathrm{Conf}_k)),$$

we can show that given a differential in the spectral sequence for Conf_k, it gives rise to infinite family of differentials in the spectral sequence for Conf_{pk}, where W_* is a free resolution over Σ_p and F_* is a free resolution over Σ_k.

4.1. **Construction.** Let

$$D_{(m,n,r)} = \mathbb{F}_p\langle x_n, y_{n+1}\rangle$$

be a filtered (graded) cochain complex over $\mathbb{F}_p$ such that $d(x) = y$ and

$$F^l(D) = \begin{cases} span(x,y) & \text{if } 0 \leq l \leq m \\ span(y) & \text{if } m+1 \leq l \leq m+r \\ 0 & \text{if } l \geq m+r+1 \end{cases}$$

where $m \leq n$ and $r > 1$.

Lemma 4.1. *The spectral sequence for the filtered complex $D_{(m,n,r)}$ is isomorphic to the universal spectral sequence $U_{(m,n,r)}$.*

Proof. The filtration on $D_{(m,n,r)}$ shows that there are only two non-trivial entries at its E_2-page, which are $E_2^{m,n-m} \cong \mathbb{F}_p\langle x\rangle$ and $E_2^{m+r,n-m-r+1} \cong \mathbb{F}_p\langle y\rangle$. Since the total complex D^* is acyclic, so the differential $d_r^{m,n-m}$ must be an isomorphism. $\square$

Through this section, for convenience, we will denote

$$D := D_{(m,n,r)}.$$

Let $C = C^{*,*}$, a double complex with horizontal differentials $d_\rightarrow$ and vertical differentials $d_\uparrow$. We will filter the total complex of C by the columns, and this produces a spectral sequence for the double complex such that the differentials d_0 are just $d_\uparrow$ at E_0 and the differentials d_1 are induced from $d_\rightarrow$. Besides, let α be an element in $E_r^{m,n-m}(C)$, the spectral sequence of the double complex C, such that

$$d_r^{m,n-m}(\alpha) = \beta \in E_r^{m+r,n-m-r+1}.$$

Lemma 4.2 ([BT13, p. 163])**.** *Assume $a \in E_0^{m,n-m}$ lives to E_r such that $[a]_r = \alpha$, i.e., a is a cocycle in $E_1, E_2, \ldots, E_{r-1}$. Then, there exists a zig-zag diagram at E_0*

$$\begin{tikzcd}
0 & & & & \\
a \arrow[u] \arrow[r] & {} & & & \\
& b_1 \arrow[u] \arrow[r] & {} & & \\
& & b_2 \arrow[u] \arrow[r] & \cdots & \\
& & & \cdots \arrow[r] & {} \\
& & & & b_{r-1} \arrow[u]
\end{tikzcd}$$

such that $d_\uparrow(a) = 0$, $d_\rightarrow(a) = d_\uparrow(b_1), \ldots, d_\rightarrow(b_{r-2}) = d_\uparrow(b_{r-1})$, and $\beta = [d_\rightarrow(b_{r-1})]_r$.

Proposition 4.3. *Let $\alpha \in E_r^{m,n-m}(C)$ and $d_r(\alpha) = \beta$, then there is a map of spectral sequences $E^{*,*}(D) \to E^{*,*}(C)$ such that $[x] \mapsto \alpha$ and $[y] \mapsto \beta$.*

Proof. Since x is the generator of $\mathbb{F}_p[n]$ and y is the generator of $\mathbb{F}_p[n+1]$ in D, and we can produce a map between two total complexes $f : D \to C$ such that

$$x \longmapsto a + b_1 + \cdots + b_{r-1}$$
$$y \longmapsto d_{\to}(b_{r-1})$$

By the previous lemma, f is a chain map since

$$(d_{\to} - d_{\uparrow})(a + b_1 \cdots + b_{r-1}) = d_{\to}(b_{r-1}).$$

By filtering C via the columns, we have

$$F^l(C) = \bigoplus_{i \geq l} \bigoplus_{q \geq 0} C^{i,q}.$$

Note that f also preserves filtrations since

$$a + b_1 + \cdots + b_{r-1} \in C^{m,n-m} \oplus C^{m+1,n-m-1} \oplus \cdots \oplus C^{m+r-1,n-m-r+1},$$

which is a summand in $F^0(C), F^1(C), \ldots, F^m(C)$. Besides,

$$d_{\to}(b_{r-1}) \in C^{m+r,n-m-r+1}$$

which is a summand in $F^0(C), F^1(C), \ldots, F^m(C), F^{m+1}(C), \ldots, F^{m+r}(C)$. Hence, this filtered chain map f induces a map between two spectral sequences. □

This filtered map from above induces a filtered map between

$$\mathrm{Hom}_{\Sigma_p}(W_*, D^{\otimes p}) \to \mathrm{Hom}_{\Sigma_p}(W_*, C^{\otimes p}),$$

where W_* is a free resolution of $\mathbb{F}_p$ over Σ_p.

Convention 4.4. (1) Given a filtration on P_*, a free resolution over some group G, such that

$$F_s(P_*) : P_s \xrightarrow{\delta} P_{s-1} \xrightarrow{\delta} \ldots \xrightarrow{\delta} P_0 \to 0,$$

there is an induced decreasing filtration on $\mathrm{Hom}(P_*, \mathbb{F}_p)$:

$$F^s(\mathrm{Hom}(P_*, \mathbb{F}_p)) : 0 \to \mathrm{Hom}(P_s, \mathbb{F}_p) \to \mathrm{Hom}(P_{s+1}, \mathbb{F}_p) \to \ldots.$$

(2) Given filtrations on two cochain complexes D_1^* and D_2^*, we have

$$F^s(D_1^* \oplus D_2^*) = F^s(D_1^*) \oplus F^s(D_2^*)$$

and

$$F^s(D_1^* \otimes D_2^*) = \sum_{s_1+s_2=s} F^{s_1}(D_1^*) \otimes F^{s_2}(D_2^*).$$

Denote:

$$\mathrm{Hom}(P_{\geq s}, \mathbb{F}_p) := F^s(\mathrm{Hom}(P_*, \mathbb{F}_p)).$$

Let $C = \mathrm{Hom}_{\Sigma_r}(V_*, M^*)$, where V_* is a free resolution over Σ_r and M^* is a cochain complex with a Σ_r-action.

Note that

$$\begin{aligned}
\mathrm{Hom}_{\Sigma_p}(W_*, C^{\otimes p}) &= \mathrm{Hom}_{\Sigma_p}(W_*, (\mathrm{Hom}_{\Sigma_r}(V_*, M^*))^{\otimes p}) \\
&\cong \mathrm{Hom}_{\Sigma_p}(W_*, (\mathrm{Hom}_{(\Sigma_r)^p}((V_*)^{\otimes p}, (M^*)^{\otimes p}))) \\
&\cong \mathrm{Hom}_{\Sigma_p \wr \Sigma_r}(W_* \otimes (V_*)^{\otimes p}, (M^*)^{\otimes p})
\end{aligned}$$

The last isomorphism is a filtration preserving map.

Lemma 4.5.

$$S := \mathrm{Hom}_{\Sigma_p}(W_*, (\mathrm{Hom}_{(\Sigma_r)^p}((V_*)^{\otimes p}, (M^*)^{\otimes p}))) \cong \mathrm{Hom}_{\Sigma_p \wr \Sigma_r}(W_* \otimes (V_*)^{\otimes p}, (M^*)^{\otimes p}) =: T$$

preserves the filtration.

Proof.

$$\begin{aligned}
F^s(T) &= \sum_{s_1+s_2=s} F^{s_1}(\mathrm{Hom}(W_*, \mathbb{F}_p)) \otimes_{\Sigma_p} F^{s_2}(\mathrm{Hom}_{(\Sigma_r)^p}((V_*)^{\otimes p}, (M^*)^{\otimes p}))) \\
&= \sum_{s_1+s_2=s} \mathrm{Hom}(W_{\geq s_1}, \mathbb{F}_p) \otimes_{\Sigma_p} \mathrm{Hom}_{(\Sigma_r)^p}(((V_*)^{\otimes p})_{\geq s_2}, (M^*)^{\otimes p})) \\
&\cong \sum_{s_1+s_2=s} \mathrm{Hom}_{\Sigma_p}(W_{\geq s_1}, \mathrm{Hom}_{(\Sigma_r)^p}(((V_*)^{\otimes p})_{\geq s_2}, (M^*)^{\otimes p})) \\
&\cong \sum_{s_1+s_2=s} \mathrm{Hom}_{\Sigma_p \wr \Sigma_r}(W_{\geq s_1} \otimes ((V_*)^{\otimes p})_{\geq s_2}, (M^*)^{\otimes p}) \\
&= \mathrm{Hom}_{\Sigma_p \wr \Sigma_r}((W_* \otimes (V_*)^{\otimes p})_s, (M^*)^{\otimes p}) \\
&= F^s(S)
\end{aligned}$$

where the third isomorphism is by the tensor-hom adjunction for chain complexes. □

4.2. **The spectral sequence of** $\mathrm{Hom}_{\Sigma_p}(W_*, D^{\otimes p})$**.** We shall compute the spectral sequence of $\mathrm{Hom}_{\Sigma_p}(W_*, D^{\otimes p})$ with given filtrations on W_* and D.

We acquire the cochain complex $D^{\otimes p}$ by performing tensor product for itself p times. We have

$$(D^{\otimes p})^{pn+l} \cong \mathrm{Ind}_{\Sigma_{p-l}\times\Sigma_l}^{\Sigma_p}(\mathbb{F}_p[n])^{\otimes p-l} \otimes (\mathbb{F}_p[n+1])^{\otimes l}$$

for $l = 0, 1, \dots, p$ at degree $pn + l$ in this complex, and the filtration is induced by the filtration on D.

Proposition 4.6. *Let W_* be a free resolution of $\mathbb{F}_p$ over Σ_p, and n be even for $D_{(m,n,r)}$. The spectral sequence for the filtered complex $\mathrm{Hom}_{\Sigma_p}\big(W_*, (D_{(m,n,r)})^{\otimes p}\big)$ is isomorphic to*

$$U_{\big(pm,pn,r\big)} \oplus \bigoplus_{t\geq 1, \varepsilon=0,1} U_{\big(pm+2t(p-1)-\varepsilon, pn+2t(p-1)-\varepsilon, p(r-1)+1\big)}.$$

Proof. The filtration on $\mathrm{Hom}_{\Sigma_p}(W_*, D^{\otimes p})$ takes the form:

$$\begin{aligned}
F^s(\mathrm{Hom}_{\Sigma_p}(W_*, D^{\otimes p})) &= \sum_{s_1+s_2=s} F^{s_1}(\mathrm{Hom}(W_*, \mathbb{F}_p)) \otimes_{\Sigma_p} F^{s_2}(D^{\otimes p}) \\
&= \sum_{s_1+s_2=s} \mathrm{Hom}_{\Sigma_p}(W_{\geq s_1}, F^{s_2}(D^{\otimes p}))
\end{aligned}$$

Since $D^{\otimes p}$ is a cochain complex with the first non-trivial degree at np,

$$\sum_{s_1+s_2=s} \mathrm{Hom}_{\Sigma_p}(W_{\geq s_1}, F^{s_2}(D^{\otimes p}))$$

is a sub-complex which is isomorphic to $\mathrm{Hom}_{\Sigma_p}(W_*, D^{\otimes p})$ for the degree $\geq np + s$ and is trivial for the degree $< np + s$. Hence, the associated graded of $\mathrm{Hom}_{\Sigma_p}(W_*, D^{\otimes p})$ is

$$F^s/F^{s+1} \cong \bigoplus_{s_1+s_2=s} \mathrm{Hom}_{\Sigma_p}(W_{s_1}, F^{s_2}(D^{\otimes p})/F^{s_2+1}(D^{\otimes p})).$$

Since $\mathrm{Hom}_{\Sigma_p}(W_{s_1}, -)$ is an exact functor, we have

$$E_1^{s,i-s} \cong H^i(F^s/F^{s+1}) \cong \bigoplus_{s_1+s_2=s} \mathrm{Hom}_{\Sigma_p}\Big(W_{s_1}, H^{i-s_1}\Big(F^{s_2}(D^{\otimes p})/F^{s_2+1}(D^{\otimes p})\Big)\Big).$$

Hence,

$$E_2^{s,i-s} \cong \bigoplus_{s_1+s_2=s} H^{s_1}\Big(\Sigma_p, H^{i-s_1}\Big(F^{s_2}(D^{\otimes p})/F^{s_2+1}(D^{\otimes p})\Big)\Big).$$

Note that $F^s(D^{\otimes p})/F^{s+1}(D^{\otimes p})$ is trivial unless $s = pm + lr$ for $0 \le l \le p$. Besides, we have

$$H^j\Big(F^{pm+lr}(D^{\otimes p})/F^{pm+lr+1}(D^{\otimes p})\Big) = \begin{cases} \mathrm{Ind}_{\Sigma_{p-l}\times\Sigma_l}^{\Sigma_p}(\mathbb{F}_p[n])^{\otimes p-l} \otimes (\mathbb{F}_p[n+1])^{\otimes l} & j = pn + l \\ 0 & \text{otherwise} \end{cases}$$

These reduce the E_2 page to

$$E_2^{s,pn+l-pm-lr} \cong H^{s-pm-lr}(\Sigma_p, \mathrm{Ind}_{\Sigma_{p-l}\times\Sigma_l}^{\Sigma_p}(\mathbb{F}_p[n])^{\otimes p-l} \otimes (\mathbb{F}_p[n+1])^{\otimes l}) \tag{4.7}$$

with other rows trivial. For $l = 0$,

$$E_2^{s,pn-pm} \cong H^{s-pm}(\Sigma_p, (\mathbb{F}_p[n])^{\otimes p}) \cong H^{s-pm}(\Sigma_p, \mathbb{F}_p).$$

By Theorem 2.15, we see that this is trivial unless $s = pm + 2t(p-1) - \varepsilon$ for $t \ge 0$ and $\varepsilon = 0, 1$. For $l = 1$,

$$E_2^{s,pn-pm-r+1} \cong H^{s-pm-r}(\Sigma_p, \mathrm{Ind}_{\Sigma_{p-1}\times\Sigma_1}^{\Sigma_p}(\mathbb{F}_p[n])^{\otimes p-1} \otimes \mathbb{F}_p[n+1]).$$

By Shapiro's lemma and Corollary 2.5, we see that this is trivial unless $s = pm + r$. For $l = 2, 3, \ldots, p-1$, it has the similar argument as the $l = 1$ case but it is trivial for every s by the invariant of the sign representation is trivial. For $l = p$,

$$E_2^{s,pn-pm-pr+p} \cong H^{s-pm-pr}(\Sigma_p, (\mathbb{F}_p[n+1])^{\otimes p}) \cong H^{s-pm-pr}(\Sigma_p, \mathbb{F}_p^{sgn}).$$

Similarly, we see that this is trivial unless $s = pm + pr + p - 1 + 2t(p-1) - \varepsilon$ for $t \ge 0$.

Hence, the E_2-page of the spectral sequence takes the form of Figure 4 after the simplification. Since $D^{\otimes p}$ is quasi-isomorphic to the zero complex, by Proposition 2.12, the infinity page $E_\infty^{s,t}$ is trivial for $s \ge 0$ and $t \ge 0$. This implies that every differential in Figure 4 (black arrows) is an isomorphism.

□

As for the case of n being odd, it is similar. See Figure 5 for its E_2-page. Since its $E_\infty^{s,t}$ is trivial for $s \ge 0$ and $t \ge 0$, this implies that the red differential d_r in Figure 5 must be a trivial map, and every black differential is an isomorphism.

Proposition 4.8. *Let W_* be a free resolution of $\mathbb{F}_p$ over Σ_p, and n be odd for $D_{(m,n,r)}$. The spectral sequence for the filtered complex $\mathrm{Hom}_{\Sigma_p}\big(W_*, (D_{(m,n,r)})^{\otimes p}\big)$ is isomorphic to*

$$\begin{aligned} &U_{\left(pm+p-2,pn+p-2,(p-1)(r-1)+1\right)} \oplus U_{\left(pm+p-1,pn+p-1,p(r-1)+1\right)} \\ &\oplus \bigoplus_{t\ge1,\varepsilon=0,1} U_{\left(pm+(2t+1)(p-1)-\varepsilon,pn+(2t+1)(p-1)-\varepsilon,p(r-1)+1\right)}. \end{aligned}$$

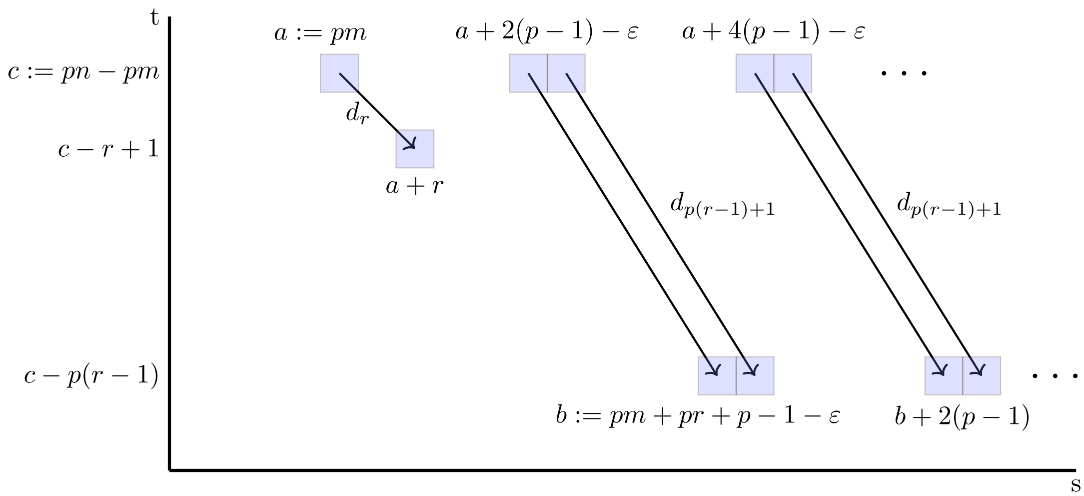

FIGURE 4. The spectral sequence of $\mathrm{Hom}_{\Sigma_p}(W_*, D^{\otimes p})$ if n is even.

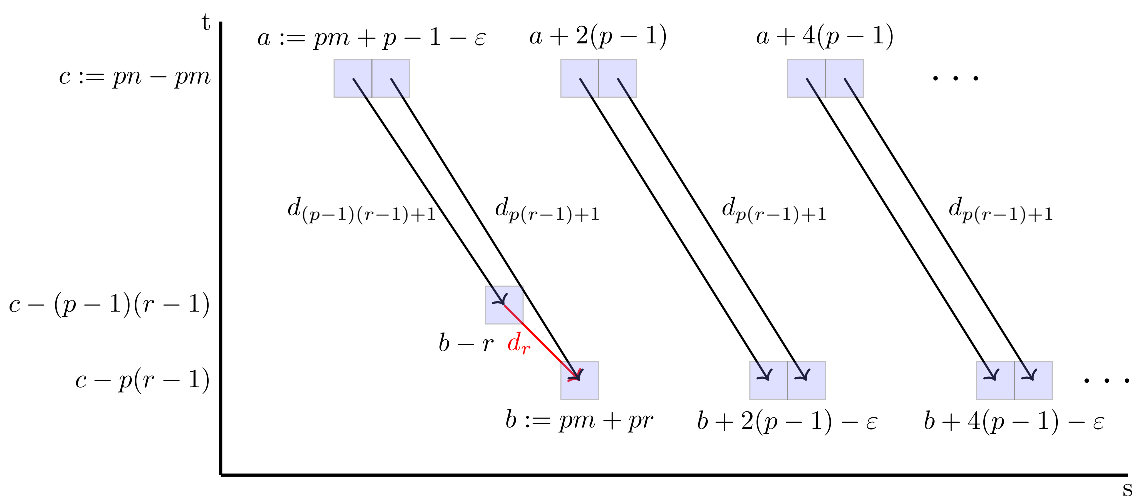

FIGURE 5. The spectral sequence of $\mathrm{Hom}_{\Sigma_p}(W_*, D^{\otimes p})$ if n is odd.

4.3. **Some related results.** We shall list some properties of the universal spectral sequences. The proof of the following statements is similar to Proposition 4.6 and Proposition 4.8.

Lemma 4.9. *Let a be an positive integer such that $a < p$ and W^a_* is a free resolution over Σ_a, then the spectral sequence for*

$$\mathrm{Hom}_{\Sigma_a}(W^a_*, (D_{(m,n,r)})^{\otimes a})$$

is isomorphic to

$$\begin{cases} U_{(am,an,r)} & \textit{if } n \textit{ is even} \\ U_{(am+(a-1)r,an+a-1,r)} & \textit{if } n \textit{ is odd} \end{cases}$$

We can also construct a filtered complex for the universal spectral sequence $V_{(m,n)}$. Let

$$S_{(m,n)} = \mathbb{F}_p\langle x_n\rangle$$

be a filtered (graded) cochain complex over $\mathbb{F}_p$ and

$$F^l(S) = \begin{cases} S_{(m,n)} & \text{if } 0 \leq l \leq m \\ 0 & \text{if } l \geq m+1 \end{cases}$$

where $m \leq n$.

Proposition 4.10. *The spectral sequence for the filtered complex $S_{(m,n)}$ is isomorphic to the universal spectral sequence $V_{(m,n)}$. Moreover, the spectral sequence for the filtered complex*

$$\mathrm{Hom}_{\Sigma_p}\big(W_*, (S_{(m,n)})^{\otimes p}\big)$$

is isomorphic to

$$\bigoplus_{t\geq 0,\varepsilon=0,1} V_{\left(pm+2t(p-1)-\varepsilon,pn+2t(p-1)-\varepsilon\right)}$$

if n is even, and is isomorphic to

$$\bigoplus_{t\geq 0,\varepsilon=0,1} V_{\left(pm+(2t+1)(p-1)-\varepsilon,pn+(2t+1)(p-1)-\varepsilon\right)}$$

if n is odd.

Lemma 4.11. *Let a be an positive integer such that $a < p$ and W^a_* is a free resolution over Σ_a, then the spectral sequence for*

$$\mathrm{Hom}_{\Sigma_a}(W^a_*, (S_{(m,n)})^{\otimes a})$$

is isomorphic to

$$\begin{cases} V_{(am,an)} & \textit{if } n \textit{ is even} \\ V_{(m,n)} & \textit{if } n \textit{ is odd and } a=1 \\ 0 & \textit{if } n \textit{ is odd and } a>1 \end{cases}$$

5. Spectral sequence for the braid group

5.1. **The framwork.** Since Cartan–Leray spectral sequence is essentially graded cochain complex, the following statement can be useful for maps between Cartan–Leray spectral sequences.

Lemma 5.1. *Let C^* be a cochain complex with differential ∂ and D^* be a cochain complex with differential d. Let $f^* : C^* \to D^* \to C^*$ be an endomorphism of a cochain complex C^* such that f^n is an isomorphism in degree n and C^n is of finite dimensional, then the induced map*

$$\ker\partial^n/\operatorname{im}\partial^{n-1} \to \ker d^n/\operatorname{im} d^{n-1} \to \ker\partial^n/\operatorname{im}\partial^{n-1}$$

is an isomorphism in degree n as well.

Proof. Since f^* is a cochain map, it restricts to an endomorphism of $\ker\partial^n$ and an endomorphism of $\operatorname{im}\partial^{n-1}$. Since f^n is injective, these two restrictions are also injective, hence isomorphism by finite dimensionality. □

Let k be a positive integer such that $k \leq 2p^2$. We define

$$\Pi_k^{\mathrm{CNT}} := \Big\{\mu = (\underbrace{2p^2, 2p^2, \ldots, 2p^2}_{b_2}, \underbrace{p^2, p^2, \ldots, p^2}_{a_2}, \underbrace{2p, 2p, \ldots, 2p}_{b_1}, \underbrace{p, p, \ldots, p}_{a_1}, \underbrace{2, 2, \ldots, 2}_{b_0 = ip+\varepsilon, i \geq 0, \varepsilon = 0,1}, \underbrace{1, 1, \ldots, 1}_{a_0}) | \mu \vdash k\Big\}.$$

and a subset of Π_k^{CNT} as

$$\hat{\Pi}_k^{\mathrm{CNT}} := \Big\{\mu = (\underbrace{2p^2, 2p^2, \ldots, 2p^2}_{b_2 < p}, \underbrace{p^2, p^2, \ldots, p^2}_{a_2 < p}, \underbrace{2p, 2p, \ldots, 2p}_{b_1 < p}, \underbrace{p, p, \ldots, p}_{a_1 < p}, \underbrace{2}_{b_0 = \varepsilon}, \underbrace{1, 1, \ldots, 1}_{a_0 < p}) | \mu \vdash k\Big\}.$$

For convenience, we let

$$C(k) = \bigoplus_{\mu \vdash k} H^*(\Sigma_k, M_\mu).$$

By Proposition 3.4, $C(k)$ is the spectral sequence of the braid group $B_k(\mathbb{R}^2)$ at E_2.

Let λ_i be a partition of $2^\varepsilon p^r$ for $0 \leq i \leq p(r-1)$ and $\varepsilon = 0, 1$ such that λ_0 is p-subordinate to $(2^\varepsilon p^r)$, λ_i is p-subordinate to λ_{i-1} for $1 \leq i \leq p(r-1)$ and $\lambda_{p(r-1)} = (\underbrace{2^\varepsilon, 2^\varepsilon, \ldots, 2^\varepsilon}_{p^r})$.

Proposition 5.2. *Let k be a positive integer such that $k \leq 2p^2$, then for all $* \geq 0$, $H^*(\Sigma_k, M_k) = 0$ unless $k = p^r$ or $2p^r$ for some $r \geq 0$. Besides, let $r \geq 0$ and $\varepsilon = 0$ or 1, then the set of partitions*

$$S_{2^\varepsilon p^r} = \{(2^\varepsilon p^r), \lambda_0, \lambda_1, \ldots, \lambda_{p(r-1)}\}$$

spans a spectral subsequence of the spectral sequence for $B_{2^\varepsilon p^r}(\mathbb{R}^2)$.

Remark 5.3. The first part of the statement is called the Arone–Mahowald theorem.

Proof. We prove the statement using the induction. The base case has been showed in Cohen's calculation for $k = p$. Without loss of generality, let $2p^{r-1} < k \leq p^r$ for some r, and we assume $H^*(\Sigma_{k'}, M_{k'}) = 0$ if $k' < k$ and k' is not equal to p^j or $2p^j$ for some j. Besides, we assume $S_1, S_2, \ldots, S_{p^{r-1}}, S_{2p^{r-1}}$ span spectral subsequences, hence there are sum of universal spectral sequences $U(1), U(2), \ldots, U(p^{r-1}), U(2p^{r-1})$ such that they provide models for $S_1, S_2, \ldots, S_{p^{r-1}}, S_{2p^{r-1}}$ respectively. By Corollary 2.6 and the first assumption above,

$$C(k) \cong \bigoplus_{\mu \in \Pi_k^{\mathrm{CNT}} \cup (k)} H^*(\Sigma_k, M_\mu).$$

Let

$$O(k) = \bigoplus_{\substack{\mu \neq (k) \\ \mu \in \hat{\Pi}_k^{\mathrm{CNT}}}} \bigotimes_{0 \leq i \leq r-1} U(p^i)^{\otimes a_i} \otimes U(2p^i)^{\otimes b_i}.$$

Fix a $\mu \in \hat{\Pi}_k^{\mathrm{CNT}}$ such that $\mu \neq (k)$. By definition, there is an injective i_μ of spectral sequences

$$\bigotimes_{0 \leq i \leq r-1} U(p^i)^{\otimes a_i} \otimes U(2p^i)^{\otimes b_i} \xrightarrow{i_\mu} \bigotimes_{0 \leq i \leq r-1} C(p^i)^{\otimes a_i} \otimes C(2p^i)^{\otimes b_i}.$$

Let $\mathrm{Conf}_k \to \mathrm{Conf}_\mu$ be a $\prod_{0 \leq i \leq r-1} (\Sigma_{p^i})^{a_i} \times (\Sigma_{2p^i})^{b_i}$-equivariant map out, then it induces a map of spectral sequence

$$\bigotimes_{0 \leq i \leq r-1} C(p^i)^{\otimes a_i} \otimes C(2p^i)^{\otimes b_i} \xrightarrow{f_\mu} C(k).$$

Let $\mathrm{Conf}_\mu \to \mathrm{Conf}_k$ be a $\prod_{0\le i\le r-1}(\Sigma_{p^i})^{a_i}\times(\Sigma_{2p^i})^{b_i}$-equivariant map in and let r_μ be a retraction of i_μ, then we have

$$C(k)\xrightarrow{g_\mu}\bigotimes_{0\le i\le r-1}C(p^i)^{\otimes a_i}\otimes C(2p^i)^{\otimes b_i}\xrightarrow{r_\mu}\bigotimes_{0\le i\le r-1}U(p^i)^{\otimes a_i}\otimes U(2p^i)^{\otimes b_i}.$$

Summing over all partitions in $\hat{\Pi}_k^{\mathrm{CNT}}\setminus\{(k)\}$ and denote $g=(\oplus r_\mu)\circ(\oplus g_\mu)$ and $f=(\oplus f_\mu)\circ(\oplus i_\mu)$. We have the composition of spectral sequences

$$f\circ g: C(k)\to O(k)\to C(k).$$

Lemma 5.4. *$f\circ g: C(k)\to O(k)\to C(k)$ is an isomorphism at E_2 restricting to the summand*

$$\begin{cases}\bigoplus_{\mu\in\Pi_k^{\mathrm{CNT}}}H^*(\Sigma_k,M_\lambda) & \text{if } 2p^{r-1}<k<p^r,\\ \bigoplus_{\mu\in\Pi_{p^r}^{\mathrm{CNT}}\setminus S_{p^r}}H^*(\Sigma_k,M_\lambda) & \text{if } k=p^r.\end{cases}$$

For $2p^{r-1}<k<p^r$, at $E_2^{*,k-1}$, we have

$$H^*(\Sigma_k,M_k)\xrightarrow{g}0\xrightarrow{f}H^*(\Sigma_k,M_k).$$

Combing with Lemma 5.1 for $\bigoplus_{\mu\in\Pi_k^{\mathrm{CNT}}}H^*(\Sigma_k,M_\lambda)$, then the commutative diagram of maps between $C(k)$ and $O(k)$ shows that the differential coming out of $E_t^{*,k-1}$ must be trivial for $t\ge 2$. Hence, $E_\infty^{*,k-1}\cong E_2^{*,k-1}\cong H^*(\Sigma_k,M_k)$. By Corollary 3.10, we have $H^*(\Sigma_k,M_k)=0$, i.e., $C(k)\cong\bigoplus_{\mu\in\Pi_k^{\mathrm{CNT}}}H^*(\Sigma_k,M_\mu)$. For $k=p^r$, the argument is similar, and it shows that the set of partitions S_{p^r} spans a spectral subsequence. □

For the rest of the paper, we shall construct these universal models for $S_{2^\varepsilon p^r}$.

5.2. **Models for the partition (p) and the partition $(2p)$.** Applying the language of universal spectral sequence, we can reformulate Cohen's calculation for $k=p$.

Proposition 5.5. *The Cartan–Leray spectral sequence for the braid group $B_p(\mathbb{R}^2)$ is isomorphic to $V_{(0,1)}\oplus U(p)$ where*

$$U(p):=V_{(0,0)}\oplus\bigoplus_{t\ge1,\varepsilon=0,1}U_{I(t,\varepsilon)}$$

provides a model for $S_p=\{(p),(\underbrace{1,1,\dots,1}_{p})\}$, and $I(t,\varepsilon)=(2t(p-1)-\varepsilon-p,2t(p-1)-\varepsilon-1,p)$.

Proof. In the spectral sequence of $B_p(\mathbb{R}^2)$, we have the isomorphic differential $d_p^{s,p-1}: H^s(\Sigma_p;M_p)\to H^{s+p}(\Sigma_p;\mathbb{F}_p)$ for $s>0$. By Example 2.19, $H^{s+p}(\Sigma_p;\mathbb{F}_p)$ is non-trivial and isomorphic to $\mathbb{F}_p$ only if $s=2t(p-1)-\varepsilon-p$ for $t\ge1$. By Proposition 4.3, for every t and ε, there is a map of spectral sequence from $U_{(m,n,r)}$ where $m=2t(p-1)-\varepsilon-p$, $n=2t(p-1)-\varepsilon-1$ and $r=p$. Besides, we have $E_2^{0,0}\cong H^0(\Sigma_p;\mathbb{F}_p)\cong V_{(0,0)}$, $E_2^{0,1}\cong H^0(\Sigma_2;M_2)\cong V_{(0,1)}$ and other entries are trivial. This leads to an isomorphism of two spectral sequences from $V_{(0,0)}\oplus V_{(0,1)}\oplus\bigoplus_{t\ge1,\varepsilon=0,1}U_{I(t,\varepsilon)}$ to the spectral sequence for $B_p(\mathbb{R}^2)$ where $I(t,\varepsilon)=(2t(p-1)-\varepsilon-p,2t(p-1)-\varepsilon-1,p)$. □

Remark 5.6. We can construct models for the partitions (1) and (2).

(1) $U(1):=V_{(0,0)}$ provides a model for $S_1=\{(1)\}$.
(2) $U(2):=V_{(0,1)}$ provides a model for $S_2=\{(2)\}$.

Proposition 5.7. *In the spectral sequence for the braid group $B_{2p}(\mathbb{R}^2)$,*

$$U(2p) := V_{(p-2,2p-2)} \oplus V_{(p-1,2p-1)} \oplus \bigoplus_{t\geq 1, \varepsilon=0,1} U_{J(t,\varepsilon)},$$

provides a model for $S_{2p} = \{(2p), (\underbrace{2,2,\ldots,2}_{p})\}$, where $J(t,\varepsilon) = \Big((2t+1)(p-1)-p-\varepsilon, (2t+1)(p-1)+p-1-\varepsilon, p\Big)$.

Proof. Proposition 5.2 shows that S_{2p} spans a spectral subsequence. Hence, to not contradict with Corollary 3.10, the differential connecting these two partitions must be an isomorphism. Specifically, we have the isomorphic differential $d_p^{s,p-1} : H^s(\Sigma_{2p}; M_{2p}) \to H^{s+p}(\Sigma_p; \mathbb{F}_p[1]^{\otimes p}) \cong H^{s+p}(\Sigma_p; \mathbb{F}_p^{sgn})$ for $s > 0$. By Example 2.19, $H^{s+p}(\Sigma_p; \mathbb{F}_p^{sgn})$ is non-trivial and isomorphic to $\mathbb{F}_p$ only if $s = (2t+1)(p-1)-\varepsilon-p$ for $t \geq 1$. Besides, $H^{p-2}(\Sigma_p, \mathbb{F}_p^{sgn}) \cong H^{p-1}(\Sigma_p, \mathbb{F}_p^{sgn}) \cong \mathbb{F}_p$ are not being hit by $d_p^{s,p-1}$. Hence, similar to Proposition 5.5, we can build a model for these two partitions. □

5.3. **The wreath product of spectral sequences for** $(\mathrm{Conf}_p)^p$**.** For the construction of a model for the partition (p^2),we should consider a $\Sigma_p \wr \Sigma_p$-equivariant maps out $\mathrm{Conf}_{p^2} \to (\mathrm{Conf}_p)^p$. Let W_* be a free resolution over Σ_p and W'_* be a free resolution over Σ_{p^2}, then we will be dealing with the following pathway for this section:

$$\mathrm{Hom}_{\Sigma_p}(W_*, D^{\otimes p}) \to \mathrm{Hom}_{\Sigma_p\wr\Sigma_p}(W_* \otimes (W_*)^{\otimes p}, C^*(\mathrm{Conf}_p)^{\otimes p}) \to \mathrm{Hom}_{\Sigma_{p^2}}(W'_*, C^*(\mathrm{Conf}_{p^2})).$$

Lemma 5.8. *The wreath product of spectral sequences for $(\mathrm{Conf}_p)^p$ is isomorphic to the spectral sequence for the filtered complex*

$$\mathrm{Hom}_{\Sigma_p}\Big(W_*, \big(S_{(0,0)} \oplus S_{(0,1)} \oplus \bigoplus_{t\geq 1,\varepsilon=0,1} D_{I(t,\varepsilon)}\big)^{\otimes p}\Big).$$

Proof. The spectral sequence for the wreath product is isomorphic to the spectral sequence for the complex

$$\mathrm{Hom}_{\Sigma_p\wr\Sigma_p}(W_* \otimes (W_*)^{\otimes p}, C^*(\mathrm{Conf}_p)^{\otimes p}),$$

which, by Lemma 4.5, is isomorphic to the spectral sequence for the complex

$$\mathrm{Hom}_{\Sigma_p}(W_*, \mathrm{Hom}_{\Sigma_p}(W_*, C^*(\mathrm{Conf}_p))^{\otimes p}).$$

Then, by Proposition 5.5, it is isomorphic to the spectral sequence for the filtered complex

$$\mathrm{Hom}_{\Sigma_p}\Big(W_*, \big(S_{(0,0)} \oplus S_{(0,1)} \oplus \bigoplus_{t\geq 1,\varepsilon=0,1} D_{I(t,\varepsilon)}\big)^{\otimes p}\Big).$$

□

We want to determine how differentials interact with one another among partitions in the wreath product of spectral sequences. First, we define some short-hand notations. For $l = 0, 1, \ldots, p$, let

$$\lambda_l := (\underbrace{p,p,\ldots,p}_{p-l}, \underbrace{1,1,\ldots,1}_{lp}), \tag{5.9}$$

be a partition of p^2 such that there are there are $p-l$ many p's and lp many 1's, and we use the same symbol to denote the corresponding summand $H^*(\Sigma_{p^2}, M_{\lambda_l})$ in the spectral sequence.

As for the wreath product of spectral sequences, let

$$\lambda_l' := (p)^{\otimes p-l} \otimes (\underbrace{1,1,\dots,1}_{p})^{\otimes l}, \tag{5.10}$$

and we use the same symbol to denote the corresponding summand $H^*(\Sigma_p \wr \Sigma_p; \mathrm{Ind}_{\Sigma_{p-l}\times\Sigma_l}^{\Sigma_p}(M_p)^{\otimes p-l} \otimes (\mathbb{F}_p)^{\otimes l})$. Since the partition λ_l is at row $(p-l)(p-1)$ in the spectral sequence, so we omit $S_{(0,1)}$. Hence, it suffices to look at the filtered complex

$$T = \mathrm{Hom}_{\Sigma_p}\Big(W_*, \big(S_{(0,0)} \oplus \bigoplus_{t\geq 1,\varepsilon=0,1} D_{I(t,\varepsilon)}\big)^{\otimes p}\Big). \tag{5.11}$$

By the universal property of Hom functor with respect to direct sum, T is isomorphic to

$$\mathrm{Hom}_{\Sigma_p}\Big(W_*, \big(S_{(0,0)}\big)^{\otimes p}\Big) \oplus \bigoplus_{t\geq 1,\varepsilon=0,1} \mathrm{Hom}_{\Sigma_p}\Big(W_*, \big(D_{I(t,\varepsilon)}\big)^{\otimes p}\Big) \oplus T',$$

where T' is the complement and its summand takes the form:

$$\mathrm{Hom}_{\Sigma_p}\Big(W_*, \bigoplus_{a+b_1+b_2+\dots=p} \mathrm{Ind}_{\Sigma_a\times\Sigma_{b_1}\times\Sigma_{b_2}\times\dots}^{\Sigma_p} S_{(0,0)}^{\otimes a} \otimes D_{I(t_1,\varepsilon_1)}^{\otimes b_{(t,\varepsilon)}} \otimes D_{I(t_2,\varepsilon_2)}^{\otimes b_2} \otimes \dots\Big).$$

The corresponding spectral sequence for T' is isomorphic to the tensor product of spectral sequences of the form from Lemma 4.9 with $E_{p+1} = \dots = E_\infty = 0$. We use the symbol $E^{*,*}(T')$ to denote its corresponding spectral sequence.

Proposition 5.12. *In the wreath product of spectral sequences for* $(\mathrm{Conf}_p)^p$*, the only non-trival summands among the* λ_l'*'s which survive to* E_∞ *are*

$$H^{2s(p-1)-\varepsilon}(\Sigma_p, H^0(\Sigma_p;\mathbb{F}_p)^{\otimes p}) \subseteq \lambda_p'$$

for $s \geq 0$, $\varepsilon = 0,1$. *Specifically,*

(1) *Let* $i = 2t(p-1) - p$ *for* $t \geq 1$, *the restriction of the differential to the summand*
$$H^{p-2}(\Sigma_p, H^i(\Sigma_p; M_p)^{\otimes p}) \subseteq \lambda_0'$$
with the image
$$H^0(\Sigma_1 \times \Sigma_{p-1}, H^i(\Sigma_p; M_p) \otimes H^{i+p}(\Sigma_p;\mathbb{F}_p)^{\otimes p-1}) \subseteq \lambda_{p-1}'$$
is an isomorphism at the $E_{(p-1)(p-1)+1}$.

(2) *Let* $i = 2t(p-1) - p$ *for* $t \geq 1$, *the restriction of the differential to the summand*
$$H^{p-1}(\Sigma_p, H^i(\Sigma_p; M_p)^{\otimes p}) \subseteq \lambda_0'$$
with the image
$$H^0(\Sigma_p, H^{i+p}(\Sigma_p;\mathbb{F}_p)^{\otimes p}) \subseteq \lambda_p'$$
is an isomorphism at $E_{p(p-1)+1}$.

(3) *Let* $i = 2t(p-1) - p, j = 2n(p-1)$ *for* $t \geq 1, n \geq 1$, *the restriction of the differential to the summand*
$$H^{p-1-\varepsilon+j}(\Sigma_p, H^i(\Sigma_p; M_p)^{\otimes p}) \subseteq \lambda_0'$$
with the image
$$H^{j-\varepsilon}(\Sigma_p, H^{i+p}(\Sigma_p;\mathbb{F}_p)^{\otimes p}) \subseteq \lambda_p'$$
is an isomorphism at $E_{p(p-1)+1}$.

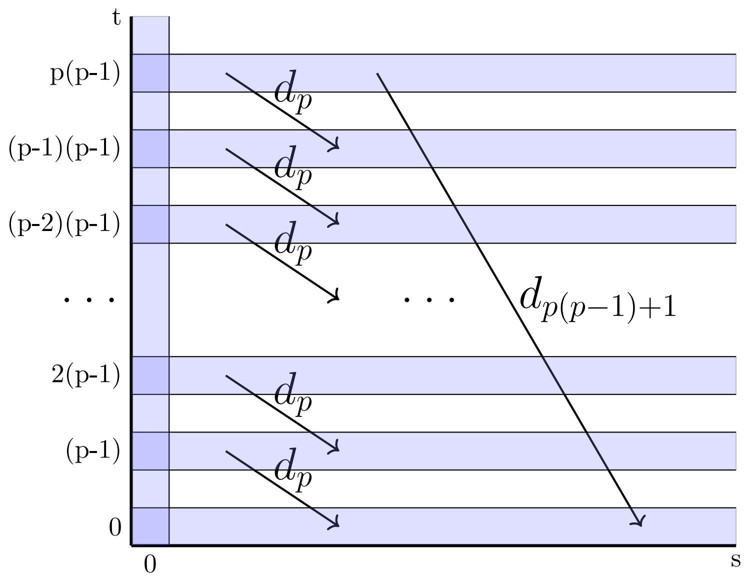

FIGURE 6. The spectral sequence for $E\Sigma_p \times_{\Sigma_p} (B_p)^p$.

(4) *Let $i = 2t(p-1) - 1 - p, j = 2n(p-1)$ for $t \geq 1, n \geq 1$, the restriction of the differential to the summand*

$$H^{j-\varepsilon}(\Sigma_p, H^i(\Sigma_p; M_p)^{\otimes p}) \subseteq \lambda'_0$$

with the image

$$H^{j-\varepsilon-(p-1)}(\Sigma_p, H^{i+p}(\Sigma_p; \mathbb{F}_p)^{\otimes p}) \subseteq \lambda'_p$$

is an isomorphism at $E_{p(p-1)+1}$.

(5) *The remaining non-trivial summands in λ'_l vanish at E_{p+1} for $l = 0, 1, 2, \ldots, p$.*

Proof. Note that by Theorem 2.14, we have

$$H^*(\Sigma_p \wr \Sigma_p; \mathrm{Ind}_{\Sigma_{p-l}\times\Sigma_l}^{\Sigma_p}(M_p)^{\otimes p-l} \otimes (\mathbb{F}_p)^{\otimes l}) \cong H^*(\Sigma_p, \mathrm{Ind}_{\Sigma_{p-l}\times\Sigma_l}^{\Sigma_p} H^*(\Sigma_p; M_p)^{\otimes p-l} \otimes H^*(\Sigma_p; \mathbb{F}_p)^{\otimes l}).$$

Hence, we should study each individual summand under the differential. $H^{2s(p-1)-\varepsilon}(\Sigma_p, H^0(\Sigma_p; \mathbb{F}_p)^{\otimes p})$ is the only summand surviving to E_∞ is due to $\mathrm{Hom}_{\Sigma_p}\left(W_*, \left(S_{(0,0)}\right)^{\otimes p}\right) \subseteq T$ and Proposition 4.10. Statement (1), (2) and (3) are by Proposition 4.8. Statement (4) is by Proposition 4.6, and Statement (5) is by the argument for the T' from above. See Figure 6 for its sketch. □

5.4. **The differential coming out of the partition** (p^2) **is injective.** We denote a_i (a'_i, respectively) to be the differential d_p (d'_p, respectively) between λ_l's (λ'_l's, respectively) at the page E_p for $i = 0, 1, 2, \ldots, p-1$. Besides, we denote d_T to be the differential coming out of the top row. Hence, the maps between spectral sequences form a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \lambda_0' \arrow[r, "a_0'"] \arrow[d, "\cong"] & \lambda_1' \arrow[r, "a_1'"] \arrow[d, "\cong"] & \dots & \dots \arrow[r, "a_{p-2}'"] & \lambda_{p-1}' \arrow[r, "a_{p-1}'"] \arrow[d, "\cong"] & \lambda_p' \arrow[r] \arrow[d, two heads, "\mathrm{cor}"] & 0 \\
0 \arrow[r] & (p^2) \arrow[r, "d_T"'] & \lambda_0 \arrow[r, "a_0"'] & \lambda_1 \arrow[r, "a_1"'] & \dots & \dots \arrow[r, "a_{p-2}"'] & \lambda_{p-1} \arrow[r, "a_{p-1}"'] & \lambda_p \arrow[r] & 0
\end{tikzcd}$$

The map between $\lambda_i' \to \lambda_i$ is an isomorphism for $i = 0, 1, \dots, p-1$ by Lemma 3.18, whereas the map between $\lambda_p' \to \lambda_p$ is a surjective map coinciding with the surjective corestriction map

$$\mathrm{cor}_H^G : H^*(\Sigma_p \wr \Sigma_p; \mathbb{F}_p) \to H^*(\Sigma_{p^2}; \mathbb{F}_p).$$

with $H = \Sigma_p \wr \Sigma_p$ and $G = \Sigma_{p^2}$.

By Theorem 2.14, we have

$$\lambda_0' = H^n(\Sigma_p \wr \Sigma_p; M_p^{\otimes p}) \cong H^0\big(\Sigma_p, H^n((\Sigma_p)^p; M_p^{\otimes p}\big) \oplus \bigoplus_{j>0} H^j\big(\Sigma_p, H^{n-j}((\Sigma_p)^p; M_p^{\otimes p}\big).$$

We define the $H^0(\Sigma_p, -)$ part of λ_0' as $H^0\big(\Sigma_p, H^n((\Sigma_p)^p; M_p^{\otimes p}\big)$, and the $H^{>0}(\Sigma_p, -)$ part of λ_0' as $\bigoplus_{j>0} H^j\big(\Sigma_p, H^{n-j}((\Sigma_p)^p; M_p^{\otimes p}\big)$. Similarly, we define the $H^0(\Sigma_p, -)$ part and the $H^{>0}(\Sigma_p, -)$ part for λ_l' and $l = 1, 2, \dots, p$. However, by Corollary 2.6, the $H^{>0}(\Sigma_p, -)$ part of λ_l' is trivial for $l = 1, 2, \dots, p-1$. Hence, the $H^0(\Sigma_p, -)$ part of λ_l' vanishes after the page E_p for $l = 0, 1, \dots, p$ except these summands from part (1) and part (2) of Proposition 5.12.

Lemma 5.13. *In the spectral sequence for the braid group $B_{p^2}(\mathbb{R}^2)$, the differential coming out of the top row*

$$d_p^{s,p^2-1} : E_p^{s,p^2-1} \cong H^s(\Sigma_{p^2}; M_{p^2}) \to E_p^{s+p,p(p-1)}$$

is injective and its image is a summand in the $H^{>0}(\Sigma_p, -)$ part of λ_0 for $s \geq 1$.

Proof. We study how the partition λ_0 vanishes in the spectral sequence for $B_{p^2}(\mathbb{R}^2)$. We can rely on the knowledge of λ_0' in the wreath product of spectral sequences. Proposition 5.12 indicates that the disappearance of λ_0' comes with three steps. The first non-trivial differential is at the page E_p such that its $E_{p+1} \cong \ker a_0' := M$ is the $H^{>0}(\Sigma_p, -)$ part of λ_0'. The second non-trivial differentials is at the page $E_{(p-1)(p-1)+1}$ such that the summands $H^{p-2}(\Sigma_p, H^i(\Sigma_p; M_p)^{\otimes p})$ vanish which are in the $H^{>0}(\Sigma_p, -)$ part of λ_0'. The last non-trivial differential is at the page $E_{p(p-1)+1}$ such that the remaining $H^{>0}(\Sigma_p, -)$ part of λ_0' vanish. We shall look at the three steps.

At the page E_p, by the diagram from above, we have

(1) $\ker a_i \cong \ker a_i'$ for $i = 0, 1, \dots, p-2$;
(2) $\mathrm{im}\, a_i \cong \mathrm{im}\, a_i'$ for $i = 0, 1, \dots p-2$;
(3) $\ker a_{p-1} \cong \ker(\mathrm{cor}_H^G \circ a_{p-1}')$;
(4) $\mathrm{im}\, a_{p-1} \cong \mathrm{im}(\mathrm{cor}_H^G \circ a_{p-1}')$.

For (3), we can further show that $\ker(\mathrm{cor}_H^G \circ a_{p-1}') \cong \ker(a_{p-1}')$ and this is because that $\mathrm{im}\, a_{p-1}'$ is a summand in the $H^0(\Sigma_p, -)$ part of λ_p' whereas $\ker(\mathrm{cor}_H^G)$ is a summand in the $H^{>0}(\Sigma_p, -)$ part of λ_p'. Let

$$L := \bigoplus_{t \geq 1} H^{p-2}(\Sigma_p, H^{2t(p-1)-p}(\Sigma_p; M_p)^{\otimes p}).$$

At the page $E_{(p-1)(p-1)+1}$, we have the following commutative diagram for the maps between two spectral sequences from λ_0 to λ_{p-1} where the partitions in the middle die at the page E_p

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, two heads, "d"] \arrow[d] & L \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & M/\operatorname{im} d_T \arrow[r, "D"'] & L \arrow[r] & 0 \end{tikzcd}$$

The diagram shows that D is surjective, so λ_{p-1} vanishes after the $E_{(p-1)(p-1)+1}$. Next, at the page $E_{p(p-1)+1}$, we have the following commutative diagram for the maps between two spectral sequences from λ_0 to λ_p

$$\begin{tikzcd} 0 \arrow[r] & \ker d \arrow[r, "d'"] \arrow[d] & \lambda'_p/\operatorname{im} a'_{p-1} \arrow[r] \arrow[d, "\operatorname{cor}_H^G"] & 0 \\ 0 \arrow[r] & \ker D \arrow[r, "D'"'] & \lambda_p/\operatorname{im} a_{p-1} \arrow[r] & 0 \end{tikzcd}$$

By Proposition 5.12,

$$N := \bigoplus_{s\geq 0,\varepsilon=0,1} H^{2s(p-1)-\varepsilon}(\Sigma_p, H^0(\Sigma_p;\mathbb{F}_p)^{\otimes p})$$

is the only non-trivial summand in λ'_p surviving to the E_∞. Hence, there exists an isomorphic copy of $\ker d$ in $\lambda'_p/\operatorname{im} a'_{p-1}$, call it K, such that

$$d' : \ker d \cong K \to \lambda'_p/\operatorname{im} a'_{p-1} \cong K \oplus N$$

is an embedding. Besides, since there does not exist an admissible set $(\varepsilon_1, s_1, \varepsilon_2, s_2)$ of length 2 such that $\varepsilon_2 = s_2 = 0$, we have $\operatorname{cor}_H^G(N) = 0$. Together this shows $\operatorname{cor}_H^G \circ d'$ is surjective, hence, D' must be surjective as well. These terms in the spectral sequence fall into the vanishing range by Corollary 3.10, and this is the last chance to vanish for $\ker D$. Hence, D' must be an isomorphism. After the page $E_{p(p-1)+1}$, every λ_i vanishes for $i = 0, 1, \ldots, p$ in the spectral sequence for the braid group. To not contradict with Corollary 3.10 again, this implies the differential d_T at the E_p must be injective. Last, d_T does not land in the $H^0(\Sigma_p, -)$ part of λ_0 is due to $\operatorname{im} d_T \subseteq \ker a_0 \cong M$. □

5.5. **A model for the partition (p^2).** The previous lemma indicates that to have a model for the partition (p^2), we need to look at the image of

$$\operatorname{Hom}_{\Sigma_p}\Big(W_*, \big(S_{(0,0)}\big)^{\otimes p}\Big)$$

and

$$\bigoplus_{t\geq 1,\varepsilon=0,1} \operatorname{Hom}_{\Sigma_p}\Big(W_*, \big(D_{I(t,\varepsilon)}\big)^{\otimes p}\Big)$$

under cor_H^G where $H = \Sigma_p \wr \Sigma_p$ and $G = \Sigma_{p^2}$ over $\mathbb{F}_p$. We keep the T' intact because T' are these $H^0(\Sigma_p, -)$ parts which shall always be mapped to their isomorphic copies in $H^*(\Sigma_{p^2}; \mathbb{F}_p)$.

Lemma 5.14. *In the spectral sequence for the braid group $B_{p^2}(\mathbb{R}^2)$,*

$$U(p^2) := V_{(0,0)} \oplus \bigoplus_{\substack{t_2\geq 1,\varepsilon_2=0,1\\ t_1\geq 0,\varepsilon_1=0,1}} U_{\widehat{I}(t_1,\varepsilon_1,t_2,\varepsilon_2)} \oplus E^{*,*}(T'),$$

provides a model for $S_{p^2} = \{(p^2), \lambda_0, \lambda_1, \ldots, \lambda_p\}$, where $\widehat{I}$ is from 5.15 and 5.16, and $E^{*,*}(T')$ is from 5.11.

Proof. Since the spectral sequence for $\mathrm{Hom}_{\Sigma_p}\Big(W_*, \big(S_{(0,0)}\big)^{\otimes p}\Big)$ is isomorphic to

$$H^{2s(p-1)-\varepsilon}(\Sigma_p, H^0(\Sigma_p;\mathbb{F}_p)^{\otimes p}) \subseteq H^*(\Sigma_p \wr \Sigma_p;\mathbb{F}_p).$$

Under $\mathrm{cor}_H^G : H^*(\Sigma_p\wr\Sigma_p;\mathbb{F}_p) \to H^*(\Sigma_{p^2};\mathbb{F}_p)$, there does not exist an admissible sequence $(\varepsilon_1, s_1, \varepsilon_2, s_2)$ such that $s_1 \geq 1$ and $\varepsilon_2 = s_2 = 0$. That is to say

$$\mathrm{cor}_H^G\Big(H^{2s(p-1)-\varepsilon}(\Sigma_p, H^0(\Sigma_p;\mathbb{F}_p)^{\otimes p})\Big) \cong \begin{cases} H^0(\Sigma_{p^2};\mathbb{F}_p) \cong \mathbb{F}_p & \text{if } s = \varepsilon = 0 \\ 0 & \text{Otherwise} \end{cases}$$

and this can be modeled by $V_{(0,0)}$ again. For

$$\bigoplus_{t_2\geq 1, \varepsilon_2=0,1} \mathrm{Hom}_{\Sigma_p}\Big(W_*, \big(D_{I(t_2,\varepsilon_2)}\big)^{\otimes p}\Big),$$

we divide this into 2 cases. For the first case, fix a $t_2 \geq 1$ and let $\varepsilon_2 = 1$, then $2t_2(p-1) - \varepsilon_2 - 1$ is even. By Proposition 4.6, the spectral sequence for $\mathrm{Hom}_{\Sigma_p}\Big(W_*, \big(D_{I(t_2,1)}\big)^{\otimes p}\Big)$ is isomorphic to

$$\bigoplus_{t_1\geq 0, \varepsilon_1=0,1} U_{\bar{I}(t_1,\varepsilon_1,t_2,1)},$$

where $\bar{I}(t_1, \varepsilon_1, t_2, 1) = \Big(p(2t_2(p-1) - 1 - p) + 2t_1(p-1) - \varepsilon_1, p(2t_2(p-1) - 2) + 2t_1(p-1) - \varepsilon_1, \bar{r}\Big)$ with

$$\bar{r} = \begin{cases} p & \text{if } t_1 = 0 \\ p(p-1)+1 & \text{if } t_1 \geq 1 \end{cases}$$

Under cor_H^G, we have that for an admissible sequence $(\varepsilon_1, s_1, 1, s_2)$, it must satisfy

$$ps_2 - 1 \geq s_1 > \frac{2s_2(p-1)-1}{2}.$$

In this case, $t_2 = s_2$ and $t_1 = s_1 - s_2(p-1) + 1$, so for fixed t_2, we have $1 \leq t_1 \leq t_2$. In other words, the summand

$$H^{2t_1(p-1)-\varepsilon_1}(\Sigma_p, H^{2t_2(p-1)-1}(\Sigma_p;\mathbb{F}_p)^{\otimes p})$$

under cor_H^G will be mapped to an isomorphic copy in $H^*(\Sigma_{p^2};\mathbb{F}_p)$ for $1 \leq t_1 \leq t_2$. Next, the rest of components in $H^*(\Sigma_p \wr \Sigma_p;\mathbb{F}_p)$ that are of the trivial images under cor_H^G, i.e., when $t_1 \geq t_2 + 1$, indicates that the components in $H^*(\Sigma_p\wr\Sigma_p;(M_p)^{\otimes p})$ which would be killed by the long differentials, i.e., $d_{(p-1)(p-1)+1}$ and $d_{p(p-1)+1}$ in Proposition 5.12, in the wreath product spectral sequence shall be killed by the differential coming from the partition (p^2) in the spectral sequence of B_{p^2} by Lemma 5.13. Hence, we can build a model to conclude all these conditions with a fix t_2 and $\varepsilon_2 = 1$:

$$U_{\widehat{I}(0,0,t_2,1)} \oplus \bigoplus_{1\leq t_1\leq t_2, \varepsilon_1=0,1} U_{\widehat{I}((t_1,\varepsilon_1,t_2,1))} \oplus \bigoplus_{t_1\geq t_2+1, \varepsilon_1=0,1} U_{\widehat{I}((t_1,\varepsilon_1,t_2,1))},$$

where

$$\widehat{I}(t_1, \varepsilon_1, t_2, 1) = \begin{cases} \bar{I}(t_1, \varepsilon_1, t_2, 1) & \text{if } 0 \leq t_1 \leq t_2 \\ (a, a + p^2 - 1, p) & \text{if } t_1 \geq t_2 + 1 \end{cases} \tag{5.15}$$

with $a := p(2t_2(p-1) - 1 - p) - p + 2t_1(p-1) - \varepsilon_1$. Note that the reason we separate $U_{\widehat{I}(0,0,t_2,1)}$ out is due to the special black differential d_r in Figure 4.

For the second case, fix a $t_2 \geq 1$ and let $\varepsilon_2 = 0$, then $2t_2(p-1)-\varepsilon_2-1$ is odd. It is similar to the even case. By Proposition 4.8, the spectral sequence for $\mathrm{Hom}_{\Sigma_p}\Big(W_*, \big(U_{I(t_2,0)}\big)^{\otimes p}\Big)$ is isomorphic to

$$\bigoplus_{t_1\geq 0, \varepsilon_1=0,1} U_{\bar{I}(t_1,\varepsilon_1,t_2,0)},$$

where $\bar{I}(t_1,\varepsilon_1,t_2,0) = \Big(p(2t_2(p-1)-p)+2t_1(p-1)+p-1-\varepsilon_1, p(2t_2(p-1)-1)+2t_1(p-1)+p-1-\varepsilon_1, \bar{r}\Big)$ with

$$\bar{r} = \begin{cases} (p-\varepsilon_1)(p-1)+1 & \text{if } t_1 = 0 \\ p(p-1)+1 & \text{if } t_1 \geq 1 \end{cases}$$

Hence, the new model takes the form:

$$U_{\widehat{I}(0,1,t_2,0)} \oplus U_{\widehat{I}(0,0,t_2,0)} \oplus \bigoplus_{1\leq t_1 \leq t_2, \varepsilon_1=0,1} U_{\widehat{I}((t_1,\varepsilon_1,t_2,0))} \oplus \bigoplus_{t_1\geq t_2+1, \varepsilon_1=0,1} U_{\widehat{I}((t_1,\varepsilon_1,t_2,0))},$$

where

$$\widehat{I}(t_1,\varepsilon_1,t_2,0) = \begin{cases} \bar{I}(t_1,\varepsilon_1,t_2,0) & \text{if } 0 \leq t_1 \leq t_2 \\ (b, b+p^2-1, p) & \text{if } t_1 \geq t_2+1 \end{cases} \tag{5.16}$$

with $b := p(2t_2(p-1)-p)-p+(2t_1+1)(p-1)-\varepsilon_1$. Note that the reason we separate $U_{\widehat{I}(0,1,t_2,0)}$ and $U_{\widehat{I}(0,0,t_2,0)}$ out is due to the first two black differentials $d_{(p-1)(r-1)+1}$ and $d_{p(r-1)+1}$ in Figure 5.

□

5.6. **A model for the partition** $(2p^2)$**.** The whole $k = 2p^2$ argument is similar to the $k = p^2$ case, where we use a $\Sigma_p \wr \Sigma_{2p}$-equivariant maps out $\mathrm{Conf}_{2p^2} \to (\mathrm{Conf}_{2p})^p$ instead.

In the spectral sequence of $k = 2p^2$, we care about the partitions $(2p^2)$, and for $l = 0, 1, \ldots, p$,

$$\mu_l := \big(\underbrace{2p, 2p, \ldots, 2p}_{p-l}, \underbrace{2, 2, \ldots, 2}_{lp}\big). \tag{5.17}$$

Let

$$D(2p) := S_{(p-2,2p-2)} \oplus S_{(p-1,2p-1)} \oplus \bigoplus_{t\geq 1, \varepsilon=0,1} D_{J(t,\varepsilon)},$$

such that the spectral sequence for $D^{(2p)}$, i.e., $U(2p)$, provides a model for the partition $(2p)$ by Proposition 5.7. Following the similar argument, it suffices to look at

$$S := \mathrm{Hom}_{\Sigma_p}(W_*, (D(2p))^{\otimes p}), \tag{5.18}$$

which is isomorphic to

$$\mathrm{Hom}_{\Sigma_p}\Big(W_*, \big(S_{(p-2,2p-2)} \oplus S_{(p-1,2p-1)}\big)^{\otimes p}\Big) \oplus \bigoplus_{t_2\geq 1, \varepsilon_2=0,1} \mathrm{Hom}_{\Sigma_p}\Big(W_*, \big(D_{J(t_2,\varepsilon_2)}\big)^{\otimes p}\Big) \oplus S',$$

where S' is the complement and its summand takes the form:

$$\mathrm{Hom}_{\Sigma_p}\Big(W_*, \bigoplus_{a_1+a_2+b_1+\cdots=p} \mathrm{Ind}_{\Sigma_{a_1}\times\Sigma_{a_2}\times\Sigma_{b_1}\times\ldots}^{\Sigma_p} S_{(p-2,2p-2)}^{\otimes a_1} \otimes S_{(p-1,2p-1)}^{\otimes a_2} \otimes D_{J(t_1,\varepsilon_1)}^{\otimes b_1} \otimes \ldots\Big).$$

Again, the corresponding spectral sequence for S' is isomorphic to the tensor product of spectral sequences of the form from Lemma 4.9 with $E_{p+1} = \cdots = E_\infty = 0$. We use the symbol $E^{*,*}(S')$ to denote its spectral sequence.

Lemma 5.19. *In the spectral sequence for the braid group $B_{2p^2}(\mathbb{R}^2)$, the differential coming out of the top row*

$$d_p^{s,2p^2-1} : E_p^{s,2p^2-1} \cong H^s(\Sigma_{2p^2}; M_{2p^2}) \to E_p^{s+p,p(2p-1)}$$

is injective and its image is a summand in the $H^{>0}(\Sigma_p, -)$ part of μ_0 for $s \geq 1$.

Proof. It is similar to Lemma 5.13 where we look at the surjective corestriction map

$$\mathrm{cor}_H^G : H^*(\Sigma_p \wr \Sigma_p; \mathbb{F}_p^{sgn}) \to H^*(\Sigma_{p^2}; \mathbb{F}_p^{sgn}),$$

with $H = \Sigma_p \wr \Sigma_p$ and $G = \Sigma_{p^2}$. □

The previous lemma indicates that to have a model for the partition $(2p^2)$, we need to look at

$$\mathrm{Hom}_{\Sigma_p}\Big(W_*, \big(S_{(p-2,2p-2)} \oplus S_{(p-12p-1)}\big)^{\otimes p}\Big)$$

and

$$\bigoplus_{t\geq 1, \varepsilon=0,1} \mathrm{Hom}_{\Sigma_p}\Big(W_*, \big(D_{J(t,\varepsilon)}\big)^{\otimes p}\Big)$$

under cor_H^G where $H = \Sigma_p \wr \Sigma_p$ and $G = \Sigma_{p^2}$ over $\mathbb{F}_p^{sgn}$. We keep the S' intact because S' are these $H^0(\Sigma_p, -)$ parts which shall always be mapped to its isomorphic copy in $H^*(\Sigma_{p^2}, \mathbb{F}_p^{sgn})$.

Lemma 5.20. *In the spectral sequence for the braid group $B_{2p^2}(\mathbb{R}^2)$,*

$$\begin{aligned} U(2p^2) := {} & V_{\big(p(p-2),p(2p-2)\big)} \oplus V_{\big((p-1)(p-1),p(2p-2)+1\big)} \\ & \oplus V_{\big(p-2+p(p-1),p-2+p(2p-1)\big)} \oplus V_{\big(p-1+p(p-1),p-1+p(2p-1)\big)} \\ & \oplus \bigoplus_{\substack{t_2\geq 1,\varepsilon_2=0,1 \\ t_1\geq 0,\varepsilon_1=0,1}} U_{\widehat{J}(t_1,\varepsilon_1,t_2,\varepsilon_2)} \oplus S' \end{aligned}$$

provides a model for $S_{2p^2} = \{(2p^2), \mu_0, \mu_1, \ldots, \mu_p\}$, where $\widehat{J}$ is from 5.21 and 5.22, and $E^{*,*}(S')$ is from 5.18.

Proof. Since $\mathrm{Hom}_{\Sigma_p}\Big(W_*, \big(S_{(p-2,2p-2)} \oplus S_{(p-12p-1)}\big)^{\otimes p}\Big)$ is isomorphic to

$$\begin{aligned} & \mathrm{Hom}_{\Sigma_p}(W_*, (S_{(p-2,2p-2)})^{\otimes p}) \oplus \mathrm{Hom}_{\Sigma_p}(W_*, (S_{(p-1,2p-1)})^{\otimes p}) \oplus \\ & \bigoplus_{\substack{a+b=p \\ a,b<p}} \mathrm{Hom}_{\Sigma_p}(W_*, \mathrm{Ind}_{\Sigma_a\times\Sigma_b}^{\Sigma_p}(S_{(p-2,2p-2)})^{\otimes a} \otimes (S_{(p-1,2p-1)})^{\otimes b}), \end{aligned}$$

we need to look at each summand individually. First, $\mathrm{Hom}_{\Sigma_p}(W_*, (S_{(p-2,2p-2)})^{\otimes p})$ is isomorphic to

$$H^{2s(p-1)-\varepsilon}(\Sigma_p, H^{p-2}(\Sigma_p; \mathbb{F}_p[1]^{\otimes p})^{\otimes p}) \subseteq H^*(\Sigma_p \wr \Sigma_p; \mathbb{F}_p^{sgn}).$$

Under $\mathrm{cor}_H^G : H^*(\Sigma_p \wr \Sigma_p; \mathbb{F}_p^{sgn}) \to H^*(\Sigma_{p^2}; \mathbb{F}_p^{sgn})$, there does not exist an admissible sequence $(\varepsilon_1, s_1, \varepsilon_2, s_2)$ such that $\varepsilon_2 = 1$ and $s_2 = 1/2$. The only term under the corestriction with non-trivial image is

$$H^0(\Sigma_p, H^{p-2}(\Sigma_p; \mathbb{F}_p[1]^{\otimes p})^{\otimes p}) \subseteq H^*(\Sigma_p \wr \Sigma_p; \mathbb{F}_p^{sgn}),$$

and this can be modeled by $V_{\big(p(p-2),p(2p-2)\big)}$. Similarly, for $\mathrm{Hom}_{\Sigma_p}(W_*, (S_{(p-1,2p-1)})^{\otimes p})$, under cor_H^G, there exists a length 2 admissible set $(\varepsilon_1, s_1, \varepsilon_2, s_2)$ such that $\varepsilon_1 = 0$ or 1, $s_1 = 5/2$, $\varepsilon_2 = 0$ and $s_2 = 1/2$. The terms under the corestriction with non-trivial image are

$$H^{p-1-\delta}(\Sigma_p, H^{p-1}(\Sigma_p; \mathbb{F}_p[1]^{\otimes p})^{\otimes p}) \subseteq H^*(\Sigma_p \wr \Sigma_p; \mathbb{F}_p^{sgn}).$$

Hence, this can be modeled by $V_{\big(p-2+p(p-1),p-2+p(2p-1)\big)} \oplus V_{\big(p-1+p(p-1),p-1+p(2p-1)\big)}$. Since the spectral sequence for $\mathrm{Hom}_{\Sigma_p}(W_*, \mathrm{Ind}_{\Sigma_a\times\Sigma_b}^{\Sigma_p}(S_{(p-2,2p-2)})^{\otimes a} \otimes (S_{(p-1,2p-1)})^{\otimes b})$ is isomorphic to the spectral sequence for the filtered complexes

$$\mathrm{Hom}_{\Sigma_a}(W_*^a, (S_{(p-2,2p-2)})^{\otimes a}) \otimes \mathrm{Hom}_{\Sigma_b}(W_*^b, (S_{(p-1,2p-1)})^{\otimes b}).$$

Since the E_2 of the second spectral sequence shall be trivial unless $b = 1$, i.e., $a = p-1$, this is saying that the spectral sequence can be modeled by $V_{\big((p-1)(p-2),(p-1)(2p-2)\big)} \otimes V_{\big(p-1,2p-1\big)} \cong V_{\big((p-1)(p-1),p(2p-2)+1\big)}$.

For

$$\bigoplus_{t_2\geq 1,\varepsilon_2=0,1} \mathrm{Hom}_{\Sigma_p}(W_*, (D_{J(t_2,\varepsilon_2)})^{\otimes p}),$$

similar to the $k = p^2$ case, we divide it into 2 cases. For the first case, fix a $t_2 \geq 1$ and let $\varepsilon_2 = 1$, then $(2t_2 + 1)(p-1) - \varepsilon_2 + p - 1$ is odd. Then, by Proposition 4.8, we have

$$\mathrm{Hom}_{\Sigma_p}\Big(W_*, \big(U_{J(t_2,1)}\big)^{\otimes p}\Big) \cong \bigoplus_{t_1\geq 0,\varepsilon_1=0,1} U_{\bar{J}(t_1,\varepsilon_1,t_2,1)},$$

where

$$\bar{J}(t_1,\varepsilon_1,t_2,1) = \Big(p((2t_2+1)(p-1)-p-1)+2t_1(p-1)+p-1-\varepsilon_1, p((2t_2+1)(p-1)+p-2)+2t_1(p-1)+p-1-\varepsilon_1, \bar{r}\Big)$$

with

$$\bar{r} = \begin{cases} (p-\varepsilon_1)(p-1)+1 & \text{if } t_1 = 0 \\ p(p-1)+1 & \text{if } t_1 \geq 1 \end{cases}$$

Under cor_H^G, we have that for an admissible sequence $(\varepsilon_1, s_1, 1, s_2)$, it must satisfy

$$ps_2 - 1 \geq s_1 > \frac{2s_2(p-1)-1}{2}.$$

In this case, $t_2 = \frac{2s_2-1}{2}$ and $t_1 = s_1 - s_2(p-1) + \frac{1}{2}$, so for fixed t_2, it results $1 \leq t_1 \leq t_2$. In other words, the summand

$$H^{2t_1(p-1)+p-1-\varepsilon_1}(\Sigma_p, H^{(2t_2+1)(p-1)-1}(\Sigma_p; \mathbb{F}_p[1]^{\otimes p})^{\otimes p})$$

under cor_H^G will be mapped to an isomorphic copy in $H^*(\Sigma_{p^2}; \mathbb{F}_p^{sgn})$ for $1 \leq t_1 \leq t_2$. Next, the rest of components in $H^*(\Sigma_p \wr \Sigma_p; \mathbb{F}_p^{sgn})$ that are of the trivial images under cor_H^G, i.e., $t_1 \geq t_2 + 1$, indicates that the components in $H^*(\Sigma_p \wr \Sigma_p; (M_{2p})^{\otimes p})$ which would be killed by the long differentials, i.e., $d_{(p-1)(p-1)+1}$ and $d_{(p(p-1)+1}$ from Proposition 5.12, in the wreath product of spectral sequences shall be killed by the differential coming from the partition $(2p^2)$ in the spectral sequence of B_{2p^2} by Lemma 5.19. Hence, we can build a model to conclude all these conditions with fixed t_2 and $\varepsilon_2 = 1$:

$$U_{\widehat{J}(0,1,t_2,1)} \oplus U_{\widehat{J}(0,0,t_2,1)} \oplus \bigoplus_{1\leq t_1\leq t_2,\varepsilon_1=0,1} U_{\widehat{J}((t_1,\varepsilon_1,t_2,1))} \oplus \bigoplus_{t_1\geq t_2+1,\varepsilon_1=0,1} U_{\widehat{J}((t_1,\varepsilon_1,t_2,1))},$$

where

$$\widehat{J}(t_1,\varepsilon_1,t_2,1)=\begin{cases}\bar{J}(t_1,\varepsilon_1,t_2,1) & \text{if } 0\le t_1\le t_2\\ (a,a+2p^2-1,p) & \text{if } t_1\ge t_2+1\end{cases} \tag{5.21}$$

and $a:=p((2t_2+1)(p-1)-p-1)-p+(2t_1+1)(p-1)-\varepsilon_1$. For the second case, fix a $t_2\ge 1$ and let $\varepsilon_2=0$, then $(2t_2+1)(p-1)+p-1$ is even. By Proposition 4.6, we have

$$\operatorname{Hom}_{\Sigma_p}\Big(W_*,\big(U_{J(t_2,0)}\big)^{\otimes p}\Big)\cong\bigoplus_{t_1\ge 0,\varepsilon_1=0,1}U_{\bar{J}(t_1,\varepsilon_1,t_2,0)},$$

where

$$\bar{J}(t_1,\varepsilon_1,t_2,0)=\Big(p((2t_2+1)(p-1)-p)+2t_1(p-1)-\varepsilon_1,p((2t_2+1)(p-1)+p-1)+2t_1(p-1)-\varepsilon_1,\bar{r}\Big)$$

with

$$\bar{r}=\begin{cases}p & \text{if } t_1=0\\ p(p-1)+1 & \text{if } t_1\ge 1\end{cases}$$

Under cor_H^G, we have that for an admissible sequence $(\varepsilon_1,s_1,1,s_2)$, it must satisfy

$$ps_2\ge s_1>s_2(p-1).$$

In this case, $t_2=\frac{2s_2-1}{2}$ and $t_1=s_1-s_2(p-1)+\frac{1}{2}$, so for fixed t_2, it results that $1\le t_1\le t_2+1$. Hence, the new model takes the form:

$$U_{\widehat{J}(0,0,t_2,0)}\oplus\bigoplus_{1\le t_1\le t_2+1,\varepsilon_1=0,1}U_{\widehat{J}((t_1,\varepsilon_1,t_2,0))}\oplus\bigoplus_{t_1\ge t_2+2,\varepsilon_1=0,1}U_{\widehat{J}((t_1,\varepsilon_1,t_2,0))},$$

where

$$\widehat{J}(t_1,\varepsilon_1,t_2,0)=\begin{cases}\bar{J}(t_1,\varepsilon_1,t_2,0) & \text{if } 0\le t_1\le t_2+1\\ (b,b+2p^2-1,p) & \text{if } t_1\ge t_2+2\end{cases} \tag{5.22}$$

and $b:=p((2t_2+1)(p-1)+p-1)-p+2t_1(p-1)-\varepsilon_1$. $\square$

5.7. **Cohomology of the braid group and Kjaer's theorem.** Since we have already constructed the models for various partition, we can define the following spectral sequences. Let

$$D(k)=\bigoplus_{\mu\in\hat{\Pi}_k^{\mathrm{CNT}}}\bigotimes_{i\ge 0}U(p^i)^{\otimes a_i}\otimes U(2p^i)^{\otimes b_i}.$$

and its invariant

$$\hat{D}(k)=\bigoplus_{\mu\in\hat{\Pi}_k^{\mathrm{CNT}}}\bigotimes_{i\ge 0}H^0\Big(\Sigma_{a_i},\big(U(p^i)\big)^{\otimes a_i}\Big)\otimes H^0\Big(\Sigma_{b_i},\big(U(2p^i)\big)^{\otimes b_i}\Big),$$

where $E_\infty(\hat{D}(k))$ is the invariant of $E_\infty(D(k))$ as well. Similar to Lemma 5.4, by summing over all partitions in $\hat{\Pi}_k^{\mathrm{CNT}}$, we have the composition of spectral sequences

$$\bar{f}\circ\bar{g}:C(k)\to D(k)\to C(k),$$

which is an isomorphism at E_2.

Theorem 5.23. *Let $k\le 2p^2$, then $\bar{f}|_{\hat{D}(k)}:\hat{D}(k)\to C(k)$ is an isomorphism of spectral sequences at E_∞.*

Proof. Fix a $\mu \in \Pi_k^{\mathrm{CNT}}$, by decomposing $U(p^i)$ and $U(2p^i)$ using universal spectral sequences V's and U's and combing terms, we can rewrite $D(k) = \bigoplus_{\mu \in \hat{\Pi}_k^{\mathrm{CNT}}} V_\mu \oplus U_\mu$. Note that at the E_2 page, by subjectivity, we have

$$E_2(D(k)) \cong \bigoplus_{\mu \in \hat{\Pi}_k^{\mathrm{CNT}}} E_2(V_\mu) \oplus E_2(U_\mu) \xrightarrow{\bar{f}_2 := \bar{f}} E_2(C(k)) \cong \bigoplus_{\mu \in \hat{\Pi}_k^{\mathrm{CNT}}} \bar{f}_2(E_2(V_\mu)) \oplus \bar{f}_2(E_2(U_\mu)).$$

By Lemma 5.1, we have the following commutative diagram at the E_r page for $r \geq 2$

$$\begin{tikzcd}
\bar{f}_r(E_r^{s,t}(U_\mu)) \arrow[r, hook, "\bar{g}_r"] \arrow[d, "d_r"] & E_r^{s,t}(U_\mu) \arrow[r, two heads, "\bar{f}_r"] \arrow[d, "d_r(U_\mu)"] & \bar{f}_r(E_r^{s,t}(U_\mu)) \arrow[d, "d_r"] \\
\bar{f}_r(E_r^{s+r,t-r+1}(U_\mu)) \arrow[r, hook, "\bar{g}_r"] & E_r^{s+r,t-r+1}(U_\mu) \arrow[r, two heads, "\bar{f}_r"] & \bar{f}_r(E_r^{s+r,t-r+1}(U_\mu))
\end{tikzcd}$$

If $d_r(U_\mu) = 0$, then $d_r = 0$. Besides, if $d_r(U_\mu)$ is an isomorphism, then d_r is also an isomorphism. As for V_μ, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
E_r^{s,t}(V_\mu) \arrow[r, two heads, "\bar{f}_r"] \arrow[d] & \bar{f}_r(E_r^{s,t}(V_\mu)) \arrow[d] \\
0 \arrow[r] & 0
\end{tikzcd}$$

Since this holds for all $r \geq 2$, we have $\bar{f}_\infty(E_\infty^{s,t}(V_\mu)) \cong \bar{f}_2(E_2^{s,t}(V_\mu))$. Combing these two arguments for V_μ and U_μ, we get that $E_\infty(C(k)) \cong \bigoplus_\mu \bar{f}_2(E_2(V_\mu))$. Besides, $\bar{f}_2$ is a composition of an isomorphism and a co-restriction map, so $E_\infty(\hat{D}(k))$, the invariant of $E_\infty(D(k))$, is isomorphic to $\bar{f}_2(E_\infty(D(k)))$. Last, we have $\bigoplus_\mu \bar{f}_2(E_2(V_\mu)) \cong \bigoplus_\mu \bar{f}_2(E_\infty(V_\mu)) \cong \bar{f}_2(E_\infty(D(k))) \cong E_\infty(\hat{D}(k))$. ∎

Corollary 5.24. *Let $l \geq 1$ such that $2lp < p^2$ and $\varepsilon = 0, 1$, then*

$$H^*(B_{p^2}(\mathbb{R}^2); \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p & \text{if } * = 0, 1, (2p-2)l, (2p-2)l+2 \\ \mathbb{F}_p \oplus \mathbb{F}_p & \text{if } * = (2p-2)l+1 \\ 0 & \text{otherwise} \end{cases}$$

Corollary 5.25. *Let $l = 1, 2, \ldots, p-1$ and $\varepsilon = 0, 1$, then*

$$H^*(B_{2p^2}(\mathbb{R}^2); \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p & \text{if } * = 0, 1, (2p-2)l, (2p-2)l+2 \\ \mathbb{F}_p \oplus \mathbb{F}_p & \text{if } * = (2p-2)l+1 \\ \mathbb{F}_p & \text{if } * = 2p^2-2p+\varepsilon, 2p^2-2+\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

Theorem 5.26. *Through degree $2p^2$, $\bigoplus_{k>0} H^*(B_k(\mathbb{R}^2); \mathbb{F}_p)$ is a free divided power algebra on generators of bidegree*

$$|x_i| = (2p^i - 2, 2p^i), |y_i| = (2p^i - 1, 2p^i)$$

for $i = 1, 2$, where the first degree is the cohomological degree and the second degree is the number of points in configuration spaces.

Theorem 5.27. *Over $\mathbb{F}_p$ and $\mathbb{R}^2$, the dimension of $H^s(\Sigma_{p^2}, M_{p^2})$ and the dimension of $H^s(\Sigma_{2p^2}, M_{2p^2})$.*

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