On sequences of iterations of increasing and continuous mappings on complete lattices

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Abstract

We generalize the famous Tarski result by showing that: if X is a complete lattice, and $f : X \to X$ is an increasing and continuous mapping, then for all points $x^0 \in X$, the limits of sequences $(f^n(\limsup_k f^k(x^0)))_{n=1}^{\infty}$ and $(f^n(\liminf_k f^k(x^0)))_{n=1}^{\infty}$ are fixed points of f. These limits are the tight fixed-point bounds between which sufficiently large iterations $f^k(x^0)$ are located. We provide an application of this result to studying best-response dynamics.

JEL Classification

C65

Key Words

Tarski's theorem, supermodular games

1 Introduction

The celebrated Tarski (1955) fixed-point theorem[1] says that an increasing (or order-preserving) mapping f on a complete lattice has a fixed point. Moreover, the set of fixed points is also a complete lattice. The lowest fixed point is the "limit" of the sequence of iterations of the lowest element of the lattice, and the highest fixed point is the "limit" of the sequence of iterations of the highest element of the lattice. In the general case, these sequences have to be transfinite, but if mapping f is in addition continuous, it suffices to study sequences indexed by natural numbers.

The Tarski theorem has numerous applications in studying discrete dynamic processes. In economics, it has also been used to prove equilibrium existence in supermodular games (see Topkis (1979) and Vives (1990)), and to prove the existence of stable matchings (Adachi (2000) and Fleiner (2003)). Echenique (2007) provides an algorithm for finding all Nash equilibria in the games of strategic complements by referring to

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[1] This result is also known as the Knaster-Tarski theorem. See Knaster and Tarski (1928). In economics, authors typically refer to it as the Tarski theorem.

arguments analogous to those yielding the Tarski theorem. Nishimura and Ok (2012) provide some other applications to optimization and games.

However, Tarski's theorem is not fully satisfying for the following reason. It says nothing about the sequences of iterations of x, when x is neither the lowest nor the highest element of a lattice. Such sequences often appear in the analysis of dynamic processes, e.g., the best-response dynamics in games. Olszewski (2020) fills the gap by showing that the properly defined limit superior and limit inferior of the sequences of iterations are fixed points of f for all x. These limits are the tight fixed-point bounds between which all sufficiently large transfinite iterations are located. Olszewski's result generalizes Tarski's theorem to all x. However, as the Tarski theorem itself, it requires using transfinite sequences.

In present paper, we show that if mapping f is in addition continuous, finding the tight fixed-point bounds does not require using any transfinite sequences. The fact that continuity lets us achieve fixed points without referring to transfinite iterations is important in applications. Indeed, the methods of achieving Nash equilibria in supermodular games (or the other equilibrium concepts in economics) via Tarski's theorem became popular partly because of its computability.[2] However, the limits superior and inferior (denoted by $\limsup_k f^k(x)$ and $\liminf_k f^k(x)$) of the sequence of iterations $(f^k(x))_{k=1}^{\infty}$ indexed by natural numbers are not necessarily fixed points of f. So, the tight fixed-point bounds for increasing and continuous mappings are obtained by taking the limits of the sequences $(f^n(\limsup_k f^k(x)))_{n=1}^{\infty}$ and $(f^n(\liminf_k f^k(x)))_{n=1}^{\infty}$, which are shown to be fixed points.

This result delivers another method of finding fixed points. As an application, we provide tight bounds in Nash equilibria for the location of players action profiles under best-response dynamics for any given initial action profile x^0. We prove this result for a large class of games, which includes what we call continuous supermodular games. Milgrom and Roberts (1990) showed that the lowest and the highest Nash equilibria are bounds for supermodular games, which in addition are uniform for all x^0. The bounds in this paper are of course tighter, usually strictly, but unlike the Milgrom and Roberts bounds they depend on x^0. In the online appendix, we propose another "application,"[3] namely, an improvement to the Echenique (2007) algorithm for finding all Nash equilibria in games of strategic complements. Unlike the Echenique algorithm, my algorithm finds not only Nash equilibria, but also all 2-period cycles in two-player games, that is, it finds all pairs of action profiles $(x_1, x_2) \neq (y_1, y_2)$ such that x_i is player i's best response to y_{-i}, and y_i is player i's best response to x_{-i} for $i = 1, 2$. The drawback of my algorithm is that it applies only to two-player games.

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[2] I am grateful to a referee for suggesting to emphasize this argument.

[3] We used quotation marks here, because the Echenique and my algorithms apply only to finite games. The results of the online appendix do not refer to Theorem 1, but they are obtained by similar arguments

2 Preliminaries

Throughout the paper $(X, \leq)$ is always a complete lattice. Completeness postulates that for all subsets $A \subseteq X$ there exist an infimum (the greatest lower bound) and a supremum (the least upper bound). They will be denoted by $\bigwedge A$ and $\bigvee A$, respectively. A monotonic (increasing or decreasing) sequence $(x^k)_{k=1}^{\infty} \subset X$ converges to an element $x^0 \in X$ if

$$x^0 = \bigvee_{k \geq 1} x^k \text{ or } x^0 = \bigwedge_{k \geq 1} x^k,$$

respectively. We then write that $\lim_k x^k = x^0$. Of course, all monotonic sequences converge by completeness. The notation $x^k \uparrow x^0$ or $x^k \downarrow x^0$ says that sequence $(x^k)_{k=1}^{\infty}$ converges to x^0, and is increasing or decreasing, respectively.

For every $x^0 \in X$, and every $f : X \to X$ define $\limsup_k f^k(x^0)$ and $\liminf_k f^k(x^0)$ as follows: First, let $x^k = f^k(x^0)$ for $k = 1, 2, ...$ be defined by induction as $f(x^{k-1})$. Then, let

$$\limsup_k f^k(x^0) = \bigwedge_{k \geq 1} \bigvee_{l \geq k} x^l$$

and

$$\liminf_k f^k(x^0) = \bigvee_{k \geq 1} \bigwedge_{l \geq k} x^l.$$

That is, $\limsup_k f^k(x^0)$ is the limit of the decreasing sequence $\left(\bigvee_{l \geq k} f^l(x^0) \right)_{k=1}^{\infty}$, and $\liminf_k f^k(x^0)$ is the limit of an increasing sequence $\left(\bigwedge_{l \geq k} f^l(x^0) \right)_{k=1}^{\infty}$.

A mapping $f : X \to Y$, where Y is another complete lattice, is *increasing* (or *order-preserving*) if $x' \leq x''$ implies that $f(x') \leq f(x'')$.

Definition 1 *Let X and Y be complete lattices. An order-preserving mapping $f : X \to Y$ is continuous if: (a) $\lim_{x^k \uparrow x} f(x^k) = f(\lim_k x^k)$ for all increasing sequences $(x^k)_{k=1}^{\infty}$; and (b) $\lim_{x^k \downarrow x} f(x^k) = f(\lim_k x^k)$ for all decreasing sequences $(x^k)_{k=1}^{\infty}$. If only part (a) of this definition is satisfied, then f is left-continuous, and if only part (b) of this definition is satisfied, then f is right-continuous.*

This definition of continuity imposes a condition on a mapping weaker than the condition imposed by Milgrom and Roberts (1990), who require the condition from our definition to hold for all chains, not only monotonic sequences. In turn, the definition used by Milgrom and Roberts imposes a weaker condition on a mapping than the definition traditionally used for mappings between lattices, which requires convergence to be preserved for all convergent nets, not only for chains. However, as Milgrom and Roberts notice, the definitions referring to chains and nets are equivalent when the domain X is a complete lattice. It is a routine to check that the Milgrom and Roberts definition is equivalent to the topological definition of continuity, which says that preimages of open sets (defined through the basis of open order intervals) are open.

The weaker concept of continuity from Definition 1 is sufficient for our main result (Theorem 1). However, when we provide Nash-equilibrium bounds for the location of players actions under the best-response dynamics in supermodular games (Theorem 2), we will need the concept of continuity as defined in Milgrom and Roberts (1990). This is so, because we will refer to one of their results, which relies on their stronger concept.

3 Main Result

The following theorem is the main result of this paper.

Theorem 1 *Suppose that* $(X, \leq)$ *is a complete lattice, and* $f : X \to X$ *is a continuous and increasing mapping. For any given* $x^0 \in X$*, let* $\overline{x}^{\omega+n} = f^n(\limsup_k f^k(x^0))$ *and* $\underline{x}^{\omega+n} = f^n(\liminf_k f^k(x^0))$ *for* $n = 1, 2, \ldots$*.*

Then: (i) sequence $(\overline{x}^{\omega+n})_{n=1}^{\infty}$ *is weakly increasing, and sequence* $(\underline{x}^{\omega+n})_{n=1}^{\infty}$ *is weakly decreasing; (ii)* $\overline{x} = \lim_n \overline{x}^{\omega+n}$ *and* $\underline{x} = \lim_n \underline{x}^{\omega+n}$ *are fixed points of* f*.*

In addition, if $\underline{y}$ *and* $\overline{y}$ *are fixed points of* f *for which there exist an increasing sequence* $(\underline{y}^k)_{k=1}^{\infty}$ *and a decreasing sequence* $(\overline{y}^k)_{k=1}^{\infty}$ *such that* $\underline{y} \leq \lim_k \underline{y}^k \leq \lim_k \overline{y}^k \leq \overline{y}$ *and* $\underline{y}^k \leq f^k(x^0) \leq \overline{y}^k$ *for all* k*, then* $\underline{y} \leq \underline{x}$ *and* $\overline{x} \leq \overline{y}$*.*

The last part from Theorem 1 captures formally the intuition that $\underline{x}$ and $\overline{x}$ are tight fixed-point bounds between which sufficiently large iterations $f^k(x^0)$ are located. This part requires some explanation. First, note that $(\underline{y}^k)_{k=1}^{\infty} = \left(\bigwedge_{l \geq k} f^l(x^0) \right)_{k=1}^{\infty}$ and $(\overline{y}^k)_{k=1}^{\infty} = \left(\bigvee_{l \geq k} f^l(x^0) \right)_{k=1}^{\infty}$ have the properties described in the last part of the theorem for $\underline{y} = \underline{x}$ and $\overline{y} = \overline{x}$. Second, one might prefer to replace the condition from the last part of Theorem 1 with the condition that for any fixed points $\underline{y}$ and $\overline{y}$,

$$\text{if } \forall_{\underline{z} < \underline{y}} \ \exists_K \ \forall_{k \geq K} \ \underline{z} < f^k(x^0), \text{ then } \underline{y} \leq \underline{x}, \tag{1}$$

and

$$\text{if } \forall_{\overline{z} > \overline{y}} \ \exists_K \ \forall_{k \geq K} \ f^k(x^0) < \overline{z}, \text{ then } \overline{x} \leq \overline{y}. \tag{2}$$

However, the problem is that this alternative condition need not be satisfied even for $\underline{y} = \underline{x}$ and $\overline{y} = \overline{x}$, because $\underline{x}$ and $\overline{x}$ (as well as $\underline{z} < \underline{x}$ and $\overline{z} > \overline{x}$) are not necessarily comparable to $f^k(x^0)$, even though $\underline{x} \leq \liminf_k f^k(x^0)$ and $\limsup_k f^k(x^0) \leq \overline{x}$. Only $\bigwedge_{l \geq k} f^l(x^0)$ and $\bigvee_{l \geq k} f^l(x^0)$ are comparable to $f^k(x^0)$. Third, conditions (1)-(2) need not imply that $\underline{y} \leq \underline{x}$ and $\overline{x} \leq \overline{y}$. They only do so if in addition: (a) $\underline{y} = \lim_n \underline{y}_n$ for some strictly increasing sequence $(\underline{y}_n)_{n=1}^{\infty}$ and $\overline{y} = \lim \overline{y}_n$ for some strictly decreasing sequence $(\overline{y}_n)_{n=1}^{\infty}$; or

(b) for every $\underline{x}'$ such that $\underline{x}' < \underline{x}$ there exist $\underline{x}''$ such that $\underline{x}' < \underline{x}'' < \underline{x}$, and for every $\overline{x}'$ such that $\overline{x} < \overline{x}'$ there exists $\overline{x}''$ such that $\overline{x} < \overline{x}'' < \overline{x}'$.[4]

Proof. We will first prove the result for $\limsup_k f^k(x^0)$. By continuity, $f(\limsup_k f^k(x^0)) = \lim_k f(\bigvee_{l\geq k} f^l(x^0))$, because the sequence $\left(\bigvee_{l\geq k} f^l(x^0)\right)_{k=1}^{\infty}$ is decreasing, and $\limsup_k x^k = \lim_k \bigvee_{l\geq k} f^l(x^0)$. Since f is increasing and $\bigvee_{l\geq k} f^l(x^0) \geq f^l(x^0)$ for $l \geq k$, we have that $f(\bigvee_{l\geq k} f^l(x^0)) \geq f(f^l(x^0)) = f^{l+1}(x^0)$ for $l \geq k$. This implies that $f(\bigvee_{l\geq k} f^l(x^0)) \geq \bigvee_{l\geq k+1} f^l(x^0) \geq \limsup_k f^k(x^0)$. Thus, $\overline{x}^{\omega+1} = f(\limsup_k f^k(x^0)) \geq \limsup_k f^k(x^0)$. (Note that in this part of the proof we only used the right-continuity of f.)

By the monotonicity of f, the sequence $(\overline{x}^{\omega+n})_{n=1}^{\infty}$ is weakly increasing, and by continuity and monotonicity $f(\overline{x}) = \lim_n f(\overline{x}^{\omega+n}) = \lim_n \overline{x}^{\omega+n+1} = \overline{x}$, so $\overline{x}$ is a fixed point of f. (Here, we used the left continuity of f.) To see the last part of the theorem, notice that $\bigvee_{l\geq k} f^l(x^0) \leq \overline{y}^k$, because the sequence $(\overline{y}^k)_{k=1}^{\infty}$ is decreasing. So, $\limsup_k f^k(x^0) \leq \lim_k \overline{y}^k \leq \overline{y}$. Since $\overline{y}$ is a fixed point of f, and f is increasing, we have that $\overline{x}^{\omega+n} \leq \overline{y}$ for all n, and so $\overline{x} \leq \overline{y}$.

We will now prove the result for $\liminf_k f^k(x^0)$. By continuity, $f(\liminf_k f^k(x^0)) = \lim_k f(\bigwedge_{l\geq k} f^l(x^0))$, because the sequence $\left(\bigwedge_{l\geq k} f^l(x^0)\right)_{k=1}^{\infty}$ is increasing, and $\liminf_k f^k(x^0) = \lim_k \bigwedge_{l\geq k} f^l(x^0)$. Since f is increasing and $\bigwedge_{l\geq k} f^l(x^0) \leq f^l(x^0)$ for $l \geq k$, we have that $f(\bigwedge_{l\geq k} f^l(x^0)) \leq f(f^l(x^0)) = f^{l+1}(x^0)$ for $l \geq k$. This implies that $f(\bigwedge_{l\geq k} f^l(x^0)) \leq \bigwedge_{l\geq k+1} f^l(x^0) \leq \liminf_k f^k(x^0)$. Thus, $\underline{x}^{\omega+1} = f(\liminf_k f^k(x^0)) \leq \liminf_k f^k(x^0)$.

By the monotonicity of f, the sequence $(\underline{x}^{\omega+n})_{n=1}^{\infty}$ is weakly decreasing, and by continuity and monotonicity $f(\underline{x}) = \lim_n f(\underline{x}^{\omega+n}) = \lim_n \underline{x}^{\omega+n+1} = \underline{x}$, so $\underline{x}$ is a fixed point of f. To see the last part of the theorem, notice that $\underline{y}^k \leq \bigwedge_{l\geq k} f^l(x^0)$, because the sequence $(\underline{y}^k)_{k=1}^{\infty}$ is increasing. So, $\underline{y} \leq \lim_k \underline{y}^k \leq \liminf_k f^k(x^0)$. Since $\underline{y}$ is a fixed point of f, and f is increasing, we have that $\underline{y} \leq \underline{x}^{\omega+n}$ for all n, and so $\underline{y} \leq \underline{x}$. ■

One may wonder whether $\overline{x} = \limsup_k f^k(x^0)$ and $\underline{x} = \liminf_k f^k(x^0)$. This is false in general as shown by the following example.

Example 1 *Let $X \subset \mathbb{R}^2$ be the sublattice of $\mathbb{R}^2$ (equipped with the coordinate-by-coordinate ordering) that*

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[4] Conditions (1)-(2) and condition (a) applied to $\underline{z} = \underline{y}_n$ and $\overline{z} = \overline{y}_n$ imply that $\underline{y}_n \leq \liminf_k f^k(x^0) \leq \limsup_k f^k(x^0) \leq \overline{y}_n$ for all n, which yields $\underline{y} \leq \liminf_k f^k(x^0) \leq \limsup_k f^k(x^0) \leq \overline{y}$. So, the result follows from the monotonicity of f, and the assumption that $\underline{y}$ and $\overline{y}$ are fixed points of f.

Condition (b) in turn implies the condition from Theorem 1. Indeed, let $\underline{x}' = \lim_k \bigwedge_{l\geq k} f^l(x^0) \wedge \underline{y}$. Then, $\underline{x}' \leq \underline{y}$. If we had $\underline{x}' < \underline{y}$, condition (1) would be violated for $\underline{z} = \underline{x}''$ from condition (b). So, $\underline{x}' = \underline{y}$, which yields a sequence $\underline{y}^k = \bigwedge_{l\geq k} f^l(x^0) \wedge \underline{y}$, $k = 1, 2, ...$ that satisfies the conditions from Theorem 1. By an analogous argument, we obtain that the sequence $\overline{y}^k = \bigvee_{l\geq k} f^l(x^0) \vee \overline{y}$, $k = 1, 2, ...$, satisfies the conditions from Theorem 1.

consists of points: $(-1,0)$, $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ *and* $(2,1)$. *(See Figure 1.) Let*

$$f(-1,0)=f(0,0)=(-1,0)\,,\; f(1,1)=f(2,1)=(2,1);$$

$$f(1,0)=(0,1) \text{ and } f(0,1)=(1,0).$$

(See again Figure 1.) The mapping f *is obviously increasing.*

Take $x^0=(0,1)$. *Then* $x^k=f^k(x)=(0,1)$ *for all even* k, *and* $x^k=f^k(x)=(1,0)$ *for all odd* k. *Thus,* $\limsup_k x^k=(1,1)$ *and* $\liminf_k x^k=(0,0)$. *However,* $f(\limsup_k x^k)=(2,1)$ *and* $f(\liminf_k x^k)=(-1,0)$, *and so* $\limsup_k x^k$ *and* $\liminf_k x^k$ *are not fixed points of* f. *Note that* $\overline{x}^{\omega+n}=(2,1)$ *and* $\underline{x}^{\omega+n}=(-1,0)$ *for* $n=1,2\dots$. *Therefore* $\overline{x}=(2,1)$ *and* $\underline{x}=(-1,0)$, *and these are fixed points of* f, *which is exactly what Theorem 1 says.*

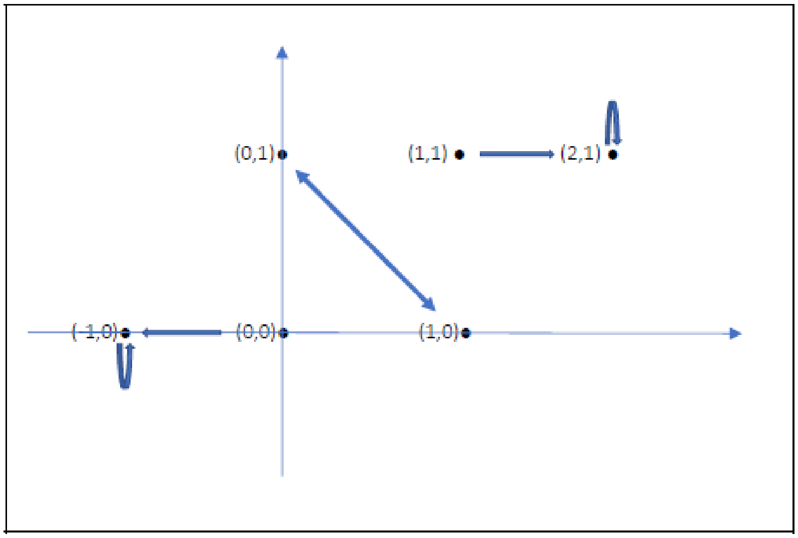

Figure 1. The lattice X described in Example 1.

Dots mark the elements of the lattice. Arrows mark mapping f.

4 Adaptive dynamics Nash-equilibrium bounds

The purpose of this section is to provide the tightest possible Nash-equilibrium bounds for the location of players' actions under best-response dynamics. We will provide these bounds for any normal-form game with the following properties:

(A) The set of actions X_i of each player i is a complete lattice.

(B) Each player i has the lowest and the highest best response to each action profile of the opponents.

Property (B) allows for defining the following two best-response mappings: $\underline{f}_i : X_{-i} \to X_i$ assigns to each $x_{-i} \in X_{-i}$ player i's action x_i that is the lowest best response to x_{-i}, and $\overline{f}_i : X_{-i} \to X_i$ assigns to each $x_{-i} \in X_{-i}$ player i's action x_i that is the highest best response to x_{-i}. In addition, equip the set of action profiles $X = X_1 \times ...X_n$ with the coordinate-by-coordinate ordering. With this ordering, X is a complete lattice. Let $\underline{f} : X \to X$ be defined by $\underline{f}(x) = (\underline{f}_1(x_{-1}), ..., \underline{f}_n(x_{-n}))$, and let $\overline{f} : X \to X$ be defined by $\overline{f}(x) = (\overline{f}_1(x_{-1}), ..., \overline{f}_n(x_{-n}))$.

(C) The mappings $\underline{f}_i$ and $\overline{f}_i$ are order-preserving, and continuous according to the Milgrom and Roberts notion of continuity (see the comment below Definition 1).

Property (C) implies that mappings $\underline{f}$ and $\overline{f}$ are also order-preserving and continuous.

Since best responses are not necessarily unique, we must specify what we mean by the best-response dynamics. Given an action profile $x^0 \in X$, we call $(x^k)_{k=0}^{\infty}$ a *best-response sequence* if x_i^{k+1} is any best response of player i to the action profile x_{-i}^k of player i's opponents.

Theorem 2 *For any given $x^0 \in X$, let $\underline{x}^k = \underline{f}^k(x^0)$ and let $\overline{x}^k = \overline{f}^k(x^0)$, for $k = 1, 2, ...$ Further, let $\underline{x}^{\omega+n} = \underline{f}^n(\liminf_k \underline{f}^k(x^0))$ and $\overline{x}^{\omega+n} = \overline{f}^n(\limsup_k \overline{f}^k(x^0))$.*

Then: (i) sequence $(\overline{x}^{\omega+n})_{n=1}^{\infty}$ is weakly increasing; (ii) sequence $(\underline{x}^{\omega+n})_{n=1}^{\infty}$ is weakly decreasing; (iii) $\overline{x} = \lim_n \overline{x}^{\omega+n}$ and $\underline{x} = \lim_n \underline{x}^{\omega+n}$ are fixed points of $\underline{f}$ and $\overline{f}$. That is, they are Nash equilibria.

In addition, suppose that $\underline{y}$ and $\overline{y}$ are Nash equilibria for which there exist: an increasing sequence $(\underline{y}^k)_{k=1}^{\infty}$ and a decreasing sequence $(\overline{y}^k)_{k=1}^{\infty}$ such that $\lim_k \underline{y}^k \geq \underline{y}$, $\lim_k \overline{y}^k \leq \overline{y}$. If for any best response sequence $(x^k)_{k=0}^{\infty}$ we have that $\underline{y}^k \leq x^k \leq \overline{y}^k$ for all k, then $\underline{y} \leq \underline{x}$ and $\overline{x} \leq \overline{y}$.

Note that $(\underline{y}^k)_{k=1}^{\infty} = \left(\bigwedge_{l \geq k} \underline{f}^l(x^0) \right)_{k=1}^{\infty}$ and $(\overline{y}^k)_{k=1}^{\infty} = \left(\bigvee_{l \geq k} \overline{f}^l(x^0) \right)_{k=1}^{\infty}$ have the properties described in the last part of the theorem for $\underline{y} = \underline{x}$ and $\overline{y} = \overline{x}$.

Proof. Theorem 2 follows from the proof of Theorem 1. The only difference between assertions (i)-(iii) in the two theorems comes from the fact that we had to use different mappings $\underline{f}$ and $\overline{f}$ for defining $\liminf_k \underline{x}^k$ and $\limsup_k \overline{x}^k$ (as well as $\underline{x}^{\omega+n}$ and $\overline{x}^{\omega+n}$) in Theorem 2, while we used the same mapping f for defining $\liminf_k x^k$ and $\limsup_k x^k$ in Theorem 1, where $x^k = f^k(x^0)$. However, Theorem 1 was proven for $\liminf_k x^k$ and for $\limsup_k x^k$ separately and independently.

The last part of Theorem 2 also follows from the proof of Theorem 1, because the condition that $\underline{y}^k \leq x^k \leq \overline{y}^k$ for any best response sequence $(x^k)_{k=0}^{\infty}$ implies (actually, is equivalent to) the condition that $\underline{y}^k \leq \underline{x}^k$ and $\overline{x}^k \leq \overline{y}^k$. ■

Milgrom and Roberts (1990) provide Nash-equilibrium bounds in supermodular games (see Theorem 8 of their paper). Their bounds apply uniformly to all initial action profiles. More specifically, they suggest starting from the lowest and the highest elements of the lattice, that is, from $\bigwedge X$ and $\bigvee X$, respectively, and consider sequences $(\underline{f}^k(\bigwedge X))_{k=1}^{\infty}$ and $(\overline{f}^k(\bigvee X))_{k=1}^{\infty}$. These sequences are monotonic (increasing and decreasing, respectively). So, they converge to Nash equilibria $\underline{z}$ and $\overline{z}$,[5] which are the Milgrom and Roberts

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[5] One can show that these are the lowest and the highest Nash equilibria.

bounds.[6]

For the definition of supermodular games, we refer the reader to their paper. Supermodular games satisfy our conditions (A) and (B). Indeed, the lowest and the highest best response exist, because the sets of best responses in a supermodular game are themselves nonempty complete lattices. (See Milgrom and Roberts (1990), Theorem 2, and Corollary from Theorem 2.[7]) By Topkis Monotonicity Theorem (quoted in Milgrom and Roberts (1990)), mappings $\underline{f}_i$ and $\overline{f}_i$ are order-preserving in supermodular games. We call a supermodular game *continuous* if it satisfies condition (C).

The Milgrom and Roberts bounds satisfy the condition from the last part of Theorem 2 with $\underline{y}^k = \underline{f}^k(\bigwedge X)$ and $\overline{y}^k = \overline{f}^k(\bigvee X)$ by the monotonicity of $\underline{f}$ and $\overline{f}$. So, the last part of Theorem 2 guarantees that $\overline{x}$ and $\underline{x}$ are tighter than the bounds from Milgrom and Roberts (1990) in the realm of continuous supermodular games. The following example shows that our bounds are strictly tighter in some games (and for some initial action profiles) than those given by Milgrom and Roberts.[8]

Example 2 *Consider the following coordination game. (This is a slightly modified version of the team of managers game studied in Milgrom and Roberts (1990), Section 4, Example (5).) Players 1 and 2 choose actions* $x_1, x_2 \in [0,1]$*, respectively.*[9] *The cost of taking action* x_i *for player* $i = 1,2$ *is* cx_i*, for some* $c \in (0,1)$*. The output of the team that consists of the two players is* $\min\{x_1, x_2\}$*. So, the payoff of player* i *is* $\min\{x_1, x_2\} - cx_i$*.*

This is obviously a continuous supermodular game. Since $c \in (0,1)$*, if player* i *takes action* x_i*, then the best response of player* i*'s opponent is taking action* x_i *as well. Therefore, the best-response dynamics can be described by the iterations of the reflection* $f : [0,1]^2 \to [0,1]^2$ *of the unit square across the diagonal* $D = \{x = (x_1, x_2) \in X : x_1 = x_2\}$ *(i.e.,* $f(x_1, x_2) = (x_2, x_1)$*).*

If $x^0 \in D$*, then* $x^k = f^k(x^0) = x^0$ *for* $k = 1, 2, \ldots$*,* $\limsup_k x^k = \liminf_k x^k = x^0$*, and* x^0 *is obviously a fixed point of* f*. If* $x^0 = (x_1^0, x_2^0) \notin D$*, then* $x^k = f^k(x^0) = x^0$ *for all even* $k = 2, 4, \ldots$*, and* $x^k = f^k(x^0) = (x_2^0, x_1^0)$ *is the image of* x^0 *under* f *for all odd* $k = 1, 3, \ldots$*.*

So, $\limsup_k x^k = x^0 \vee f(x^0)$ *is the join, and* $\liminf_k x^k = x^0 \wedge f(x^0)$ *is the meet of* x^0 *and* $f(x^0)$*. Both points are on the diagonal, and so are fixed points of* f*. The former point has both coordinates equal to* $\max\{x_1^0, x_2^0\}$*, and the latter point has both coordinates equal to* $\min\{x_1^0, x_2^0\}$*. Thus, Theorem 1 (as well as Theorem 2) imply that*

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[6] Milgrom and Roberts' bounds apply uniformly to a larger class of adaptive dynamics. Our result can also be extended to their larger class. However, this result is beyond the scope of this paper.

[7] This result relies on the stronger notion of continuity, defined as in Milgrom and Roberts (1990).

[8] Recall, however, that their bounds apply uniformly to all x^0, while the bounds from Theorem 2 depend on x^0. Since all these bounds are in Nash equilibria, the Milgrom and Roberts bounds must be the tightest possible among those that apply uniformly to all x^0.

[9] The generalization of the game and our result to more than two players is straightforward.

$$\min\{x_1^0, x_2^0\} \le x_i^k \le \max\{x_1^0, x_2^0\}$$

for $k = 1, 2, ...$ and $i = 1, 2$. In contrast, Theorem 8 in Milgrom and Roberts (1990) does not impose any bound on any x^k in this game, that is, it only says that $0 \le x_1^k, x_2^k \le 1$.

One may wonder whether the Nash-equilibrium bounds from Theorem 2 apply to all (not necessarily continuous) supermodular games. The example below shows that this conjecture is false. One can construct tight Nash-equilibrium bounds for all normal-form games that satisfy conditions (A), (B), and (C) except continuity (which include all supermodular games) by applying the construction from Olszewski (2020). However, that construction refers to transfinite induction, which makes it less useful in applications.

Example 3 *This will be a modified version of the game from Example 2. Players 1 and 2 choose actions*

$$x_1, x_2 \in \left\{\frac{n}{n+1} : n = 0, 1, ...\right\} \cup \left\{1 + \frac{n}{n+1} : n = 0, 1, ...\right\} \cup \{2\} \cup \{3\},$$

respectively. Notice that the action sets are well ordered, that is, each action x_i has the lowest higher action x_i'. The marginal cost of raising action x_i to action x_i' is either 0 or $c(x_i' - x_i)$, for some $c \in (0, 1)$. It is 0 for player 1 when $n = 1, 3, ...$ is odd, and it is 0 for player 2 when $n = 0, 2, ...$ is even. It is also 0 for both players i when $x_i = 2$. In all other cases, it is $c(x_i' - x_i)$. As in Example 2, the output of the team that consists of players 1 and 2 is $\min\{x_1, x_2\}$. So, the payoff of player i is $\min\{x_1, x_2\}-$ the cost of taking action x_i.

Checking supermodularity is straightforward.

Let $x^0 = (0, 0)$. Then, the unique best response of player 1 to the action of player 2 is 0. Player 2 has two best responses: 0 and 1/2, because the cost of raising her action from 0 to 1/2 is 0 for player 2. Thus, $\overline{f}(x^0) = (0, 1/2)$. Now, player 2 still has the same two best responses, but player 1 also has two best responses: 1/2 and 2/3, because the cost of raising her action from 1/2 to 2/3 is 0 for player 1.Thus, $\overline{f}^2(x^0) = (2/3, 1/2)$. Continuing in this manner, we obtain that

$$\overline{f}^n(x^0) = \left(\frac{n}{n+1}, \frac{n-1}{n}\right) \text{ for all even } n,$$

and

$$\overline{f}^n(x^0) = \left(\frac{n-1}{n}, \frac{n}{n+1}\right) \text{ for all odd } n.$$

Therefore, $\overline{x}^\omega = \limsup_k \overline{f}^k(x^0) = (1, 1)$.

By an analogous argument, we obtain that $\overline{x}^{\omega+1} = \overline{f}(\overline{x}^\omega) = (1, 3/2)$, $\overline{x}^{\omega+2} = \overline{f}^2(\overline{x}^\omega) = (5/3, 3/2)$, and more generally,

$$\overline{x}^{\omega+n} = \left(1 + \frac{n}{n+1}, 1 + \frac{n-1}{n}\right) \text{ for all even } n,$$

and

$$\overline{x}^{\omega+n} = \left(1 + \frac{n-1}{n}, 1 + \frac{n}{n+1}\right) \text{ for all odd } n.$$

Therefore $\overline{x} = \lim_n \overline{x}^{\omega+n} = (2,2)$*, and this is not a fixed point of* $\overline{f}$*. Indeed, since the cost of raising their actions from 2 to 3 is 0 for both players,* $\overline{f}(2,2) = (3,3)$*.*

Remark 1 *Milgrom and Roberts in their definition of supermodular games assume that each player's payoff is continuous with respect to the action profile of the opponents (for any given action of the player), and is upper semi-continuous with respect to the player's action (for any given action profile of the opponents). The example above shows that the bounds from Theorem 2 do not apply even to supermodular games in which players' payoffs are continuous functions of their action profile. The continuity of payoffs yields only the left-continuity of* $\underline{f}_i$ *and the right continuity of* $\overline{f}_i$*.*

5 References

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Online Appendix

Echenique algorithm for finding Nash equilibria

Echenique (2007) provides an algorithm for finding all Nash equilibria in games of strategic complements. We refer the reader to Echenique's paper for the definition of games of strategic complements. Only the following features of this class of games are essential for his and our analysis:

(D) The action space X_i of each player i is a finite sublattice of an Euclidean space with the coordinate-by-coordinate ordering. To simplify notation Echenique considers a specific sublattice $\{1, 2, ..., k_i\}^d \subset \mathbb{R}^d$. We will simplify notation even further by assuming that $d = 1$. All his and my results hold for arbitrary finite sublattices.

(E) For any player and any action profile of the player's opponents, there exist the highest best response and the lowest best response of the player. In addition, they are weakly increasing functions of the opponents' action profile.

(F) The feature from (E) applies to the game obtained from the original game by reducing the set of actions of each player i to an arbitrary rectangle $R_i \subset \{1, 2, ..., k_i\}^d$.

The Echenique algorithm is defined as follows:

(1) Start with the lowest action profile $x^0 = (x_1^0, ..., x_n^0)$ of the lattice $X_1 \times ... \times X_n$ (equipped with the coordinate-by-coordinate ordering). Define x_i^{k+1} as the lowest best response of player i to action profile x_{-i}^k of player i's opponents. The sequence $\{x^k\}_{k=0}^{\infty}$, where $x^k = (x_1^k, ..., x_n^k)$, is increasing by property (E). So, it becomes constant from some k, and x^k for this k is the lowest Nash equilibrium of the game.[10]

(2.i) Apply the instructions from (1) to $(x_i^k + 1, x_{-i}^k)$ (as x^0) in the game in which the set of player i's actions is restricted to the actions no smaller than $x_i^k + 1$, and the set of actions of player i's opponent is restricted to the actions no smaller than x_{-i}^k.

(3) Check if the equilibria of the games with the restricted sets of actions that were found in (2.i), for $i = 1, ..., n$, are the equilibria of the original game. Echenique shows that in order to do so one must only check if x_i^k is not a profitable deviation of player i from the action profile found in (2.i).

(4) If an equilibrium $(y_1, ..., y_n)$ of the original game was found in (2.i), examine the games in which the set of actions of some player j is restricted to the actions no lower than $y_j + 1$, and the actions of all other players j are no lower than y_j. Otherwise examine the games in which the set of actions of some player j is restricted to the actions higher at least by one than that in $(x_i^k + 1, x_{-i}^k)$ and the actions of any other player j are restricted to the actions no lower than that in $(x_i^k + 1, x_{-i}^k)$. Continue in this fashion by passing to games with more and more restricted sets of actions.

The following example illustrates the Echenique algorithm.

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[10] Alternatively, one can start from the highest action profile, and conclude Step 1 by finding the highest Nash equilibrium of the game. We will no longer define this alternative version of the Echenique algorithm.

Example 4 *Consider the game described in Example 2, but with a discrete action space* $x_1, x_2 \in \{0, 1, ..., 9\}$*, and the cost of taking action* x_i *being* $x_i^2/4$ *(instead of* cx_i *for some* $c \in (0,1)$*).*

The Echenique algorithm begins with $(x_1^0, x_2^0) = (0,0)$*. This turns our to be a Nash equilibrium. So, the algorithms begins next with* $(x_1^0, x_2^0) = (0,1)$ *or with* $(x_1^0, x_2^0) = (1,0)$*. Take the former action profile, but by the symmetry of the game the argument is analogous for the latter one. In this case, player 2 is allowed to take only actions* $x_2 \geq 1$*. The best response of player 1 to* $x_2^0 = 1$ *is* $x_1^1 = 1$*, and the best response of player 2 to* $x_1^0 = 0$ *is* $x_2^1 = 1$*, because of the restriction on the set of allowed actions. Clearly,* $x_1^2 = x_2^2 = 1$*, so* $(x_1, x_2) = (1,1)$ *is a Nash equilibrium of the game with the restricted set of actions for player 2. The next step of the Echenique algorithms is to check if* $x_2 = 0$ *is a better action for player 2 than* $x_2 = 1$*. It is not, so* $(x_1, x_2) = (1,1)$ *is a Nash equilibrium of the original game. Similarly, we find that* $(x_1, x_2) = (1,1)$ *is a Nash equilibrium of the original game by starting at* $(x_1^0, x_2^0) = (1,0)$*.*

Next, we consider action profiles $(x_1^0, x_2^0) = (0,2)$ *and* $(2,0)$ *in the games in which the set of each player's actions is restricted to the actions no smaller than the action in the given action profile. we find that* $(1,2)$ *and* $(2,1)$ *are Nash equilibria of the games with such restricted sets of actions. Thus, according to the Echenique algorithm, we must check if the action of player 2 lower by one in the former case, and the action of player 1 lower by one in the latter case are not profitable deviations in the original game. They are profitable deviations. This shows that* $(x_1^0, x_2^0) = (1,2)$ *and* $(2,1)$ *are not Nash equilibria in the original game.*

We consider next action profiles $(x_1^0, x_2^0) = (0,3), (1,2), (2,1),$ *and* $(3,0)$*. By iterating best responses, and checking if the actions lower by one are not profitable deviations, we determine that none of these four action profiles is a Nash equilibria of the original game. And so on. We conclude that only* $(0,0)$ *and* $(1,1)$ *are Nash equilibria.*

For the game from this example, the Echenique algorithm requires checking the restricted sets of allowed actions $\{(x_1, x_2) \in X_1 \times X_2 : x_1 \geq x_1^0 \text{ and } x_2 \geq x_2^0\}$ *for all* (x_1^0, x_2^0)*. Therefore, the improvement of the Echenique algorithm over the trivial algorithm of checking directly whether each action profile satisfies the equilibrium conditions comes from a smaller number of actions that had to be examined as possibly profitable deviations. For some other games, the improvement is more substantial (see Echenique (2007) for details).*

Below, we suggest a modification of the Echenique algorithm for two-player games. The advantage of my modification over the original algorithm is that it finds not only all Nash equilibria (in the games of strategic complements), but also all 2-period cycles of the best-response dynamics, which are defined as follows:

Definition 2 *A pair of action profiles* $(x_1, x_2) \neq (y_1, y_2)$ *is called a 2-period cycle if* x_i *is player i's best response to* y_{-i}*, and* y_i *is player i's best response to* x_{-i} *for* $i = 1, 2$*.*

In addition, the algorithm discovers an interesting property of the two-player games of strategic complements, namely, if (x_1, x_2) and (y_1, y_2) is a 2-period cycle, then $(\min\{x_1, y_1\}, \min\{x_2, y_2\})$ and $(\max\{x_1, y_1\}, \max\{x_2, y_2\})$ are Nash equilibria. The drawback of my algorithm is that its interesting properties hold only for two-player games. Finally, since the Echenique and my algorithms apply only to finite games, the results of this section are clearly not direct applications of Theorem 1, but they are established by referring to similar arguments.

The algorithm:

(1) Start with the action profile $x^0 = (x_1^0, x_2^0)$ such that x_1^0 is the lowest element of the lattice X_1, and x_2^0 is the highest element of the lattice X_2 (or the other way round, start with the highest and the lowest element, respectively). Define x_i^{k+1} as a specific best response of player i to action x_{-i}^k of player $-i$. This is the lowest best response of player $i = 1$ and the highest best response of player $i = 2$ in even periods, and the other way round in odd periods. The sequences $(x_1^k)_{k=0}^{\infty}$ and $(x_2^k)_{k=0}^{\infty}$ alternate between higher and lower actions. More precisely, we will show later that:

Lemma 1 *(a)* $x_1^0 \leq x_1^2 \leq x_1^4 \leq ... \leq ... \leq x_1^5 \leq x_1^3 \leq x_1^1$*;*

(b) $x_2^0 \geq x_2^2 \geq x_2^4 \geq ... \geq ... \geq x_2^5 \geq x_2^3 \geq x_2^1$*.*

Moreover, if $\{y^k\}_{k=0}^{\infty}$ is the sequence constructed by starting from any $y^0 = (y_1^0, y_2^0)$, and taking as y_i^{k+1}, for $i = 1, 2$, any best response of player i to action y_{-i}^k of player $-i$, then we will show later that:

Lemma 2 *(a)* $y_1^k \leq x_1^k$ *and* $y_2^k \geq x_2^k$ *for all odd* k*;*

(b) $y_1^k \geq x_1^k$ *and* $y_2^k \leq x_2^k$ *for all even* k*.*

Since the number of action profiles is finite, it follows from Lemma 2 that there is a K such that $(x_1^k, x_2^k) = (x_1^{k+2}, x_2^{k+2})$ for all $k \geq K$.

Lemma 3 *(a) If* $(x_1^K, x_2^K) = (x_1^{K+1}, x_2^{K+1})$*, then* (x_1^K, x_2^K) *is a Nash equilibrium of the game.*

(b) If $(x_1^K, x_2^K) \neq (x_1^{K+1}, x_2^{K+1})$*, then* (x_1^K, x_2^K) *and* (x_1^{K+1}, x_2^{K+1}) *is a 2-period cycle. In this case,* $(\min\{x_1^K, x_1^{K+1}\}, \min\{x_2^K, x_2^{K+1}\})$ *and* $(\max\{x_1^K, x_1^{K+1}\}, \max\{x_2^K, x_2^{K+1}\})$ *are Nash equilibria.*

Moreover, it follows from Lemma 3 that any Nash equilibrium or the two action profiles of any 2-period cycle must be contained in the square $[\min\{x_1^K, x_1^{K+1}\}, \max\{x_1^K, x_1^{K+1}\}] \times [\min\{x_2^K, x_2^{K+1}\}, \max\{x_2^K, x_2^{K+1}\}]$. In particular, if $(x_1^K, x_2^K) = (x_1^{K+1}, x_2^{K+1})$, then it is a unique Nash equilibrium of the game, and the game has no 2-period cycles in this case. If $(x_1^K, x_2^K) \neq (x_1^{K+1}, x_2^{K+1})$, then $(\min\{x_1^K, x_1^{K+1}\}, \min\{x_2^K, x_2^{K+1}\})$ is the lowest Nash equilibrium, and $(\max\{x_1^K, x_1^{K+1}\}, \max\{x_2^K, x_2^{K+1}\})$ is the highest Nash equilibrium.

Suppose now that $(x_1^K, x_2^K) \neq (x_1^{K+1}, x_2^{K+1})$, and that $\min\{x_1^K, x_1^{K+1}\} = x_1^K$. Then $\max\{x_1^K, x_1^{K+1}\} = x_1^{K+1}$, $\min\{x_2^K, x_2^{K+1}\} = x_2^{K+1}$, and $\max\{x_2^K, x_1^{K+1}\} = x_2^K$.

(2.1) Apply the instructions from (1) to $(x_1^K + 1, x_2^K)$ as x^0 in the game in which the set of action profiles is restricted to $[\min\{x_1^K, x_1^{K+1}\} + 1, \max\{x_1^K, x_1^{K+1}\}] \times [\min\{x_2^K, x_2^{K+1}\}, \max\{x_2^K, x_2^{K+1}\}]$.

(2.2) Apply the instructions from (1) to $(x_1^K, x_2^K - 1)$ as x^0 in the game in which the set of action profiles is restricted to $[\min\{x_1^K, x_1^{K+1}\}, \max\{x_1^K, x_1^{K+1}\}] \times [\min\{x_2^K, x_2^{K+1}\}, \max\{x_2^K, x_2^{K+1}\} - 1]$.

(3) Check if the equilibria or 2-period cycles of the games with the restricted sets actions, which were found in (2.1) and (2.2) are the equilibria or 2-period cycles, respectively, of the original game.

To do so, one must only check if x_1^K is not a profitable deviation of player 1 in (2.1) form the equilibrium or from one of the two profiles of the 2-period cycle of the game with the restricted sets of actions, and similarly if x_2^K is not a profitable deviation of player 2 in (2.2). This follows from the fact that if

$$(x_1, x_2) \in [\min\{x_1^K, x_1^{K+1}\}, \max\{x_1^K, x_1^{K+1}\}] \times [\min\{x_2^K, x_2^{K+1}\}, \max\{x_2^K, x_2^{K+1}\}],$$

then

$$(y_1, y_2) \in [\min\{x_1^K, x_1^{K+1}\}, \max\{x_1^K, x_1^{K+1}\}] \times [\min\{x_2^K, x_2^{K+1}\}, \max\{x_2^K, x_2^{K+1}\}]$$

for all best responses y_1 to x_2 and all best responses y_2 to x_1.

(4) If an equilibrium $(y_1', y_2') = (y_1'', y_2'')$ or a 2-period cycle $(y_1', y_2') \neq (y_1'', y_2'')$ of the original game was found in (2.i), i=1,2, examine the games in which either the set of actions of player 1 is restricted to the actions no lower than $\min\{y_1', y_1''\} + 1$, or the set of actions of player 2 is restricted to the actions no higher than $\max\{y_2', y_2''\} - 1$. Otherwise examine the games in which the set of actions of player 1 is restricted to the actions higher than $x_1^K + 1$ in (2.1) or to the actions higher than x_1^K in (2.2), while the set of actions of player 2 is restricted as in (2.i); and examine the games in which the set of actions of player 2 is restricted to the actions lower than x_2^K in (2.1) or to the actions lower than $x_2^K - 1$ in (2.2), while the set of actions of player 1 is restricted as in (2.i). Continue in this fashion by passing to games with smaller and smaller sets of actions.

The following example illustrates my algorithm.

Example 5 *Consider the game from Example 4. As suggested by my algorithm, begin with* $(x_1^0, x_2^0) = (0, 9)$*. The profile of best responses to this action profile is* $(1, 0)$*. Indeed, player 1 maximizes* $\min\{x_1, 9\} - x_1^2/4$*, and player 2 maximizes* $-x_2^2/4$*. Thus,* $(x_1^1, x_2^1) = (1, 0)$*. Now, player 1 maximizes* $-x_1^2/4$*, and player 2 maximizes* $\min\{1, x_2\} - x_2^2/4$*. This yields* $(x_1^2, x_2^2) = (0, 1)$*. Continuing in this manner, we obtain* $(x_1^{2k-1}, x_2^{2k-1}) = (1, 0)$ *and* $(x_1^{2k}, x_2^{2k}) = (0, 1)$ *for* $k = 1, 2, \ldots$*. Thus,* $(1, 0)$ *and* $(0, 1)$ *is a 2-period cycle of this game. By Lemma 4, action profiles* $(0, 0)$ *and* $(1, 1)$ *are Nash equilibria.*

We conclude that the game has two Nash equilibria: $(0, 0)$*,* $(1, 1)$*, and one 2-period cycle comprising action profiles* $(1, 0)$ *and* $(0, 1)$*. Of course, the example was selected purposely so that applying my algorithm is much simpler than applying the Echenique algorithm. We make no general claim that would compare the two algorithms in terms of simplicity.*

Remark 2 *It may be worth pointing out that starting at* x^0 *from step (1) of my algorithm, the action profile converges to some Nash equilibrium in the two-player games of strategic complements (or more generally the games satisfying conditions (D)-(F)) if players alternate in updating their actions. More precisely, let*

(x_1^0, x_2^0) be such that x_1^0 is the lowest element of the lattice X_1, and x_2^0 is the highest element of the lattice X_2 (or the other way round, the highest and the lowest element, respectively). For odd k's, let $x_2^k = x_2^{k-1}$, and let x_1^k be the lowest (highest) best response of player 1 to x_2^{k-1}. For even k"s, let $x_1^k = x_1^{k-1}$, and let x_2^k be the lowest (highest, respectively) best response of player 2 to x_1^{k-1}. Then, there exists a K such that $(x_1^K, x_2^K) = (x_1^{K+1}, x_2^{K+1})$, which means that (x_1^K, x_2^K) is a Nash equilibrium.

Indeed, $x_2^2 \leq x_2^0$, because x_2^0 is the highest element of X_2. This implies $x_1^3 \leq x_1^1$, because x_1^1 is the lowest best response of player 1 to x_2^0, and x_1^3 is the lowest best response of player 1 to x_2^2. This in turn implies that $x_2^4 \leq x_2^2$, because x_2^2 is the lowest best response of player 2 to x_1^1, and x_2^4 is the lowest best response of player 2 to x_1^3. Continuing in this fashion, we obtain that $... \leq x_1^5 \leq x_1^3 \leq x_1^1$ and $... \leq x_2^4 \leq x_2^2 \leq x_2^0$, and this means that the sequence $((x_1^k, x_2^k))_{k=1}^{\infty}$ must stabilize.

Finally, we provide the proofs of Lemmas 2-4.

Proof. (of Lemma 2) Since x_1^0 is the lowest element of the lattice X_1, and x_2^0 is the highest element of the lattice X_2, we have that $x_1^0 \leq x_1^1$ and $x_2^0 \geq x_2^1$. Suppose that $x_1^0 \leq x_1^2 \leq ... \leq x_1^{2k} \leq x_1^{2k+1} \leq ... \leq x_1^3 \leq x_1^1$ and $x_2^0 \geq x_2^2 \geq ... \geq x_2^{2k} \geq x_2^{2k+1} \geq ... \geq x_2^3 \geq x_2^1$ for some $k \geq 0$. Then, by virtue of (E), $x_1^{2k-2} \leq x_1^{2k} \leq x_1^{2k+1}$ implies that $x_2^{2k+2} \geq x_2^{2k+1} \geq x_2^{2k-1}$ and $x_2^{2k} \geq x_2^{2k+1} \geq x_2^{2k-1}$ implies that $x_1^{2k} \leq x_1^{2k+2} \leq x_1^{2k+1}$. The argument establishing the inequality $x_2^{2k+2} \geq x_2^{2k+1}$ also refers to the construction of the sequence $\{x^k\}_{k=0}^{\infty}$, namely, that x_2^{2k+2} is the highest best response to x_1^{2k+1}, and that x_2^{2k+1} is the lowest best response to x_1^{2k}. Similarly, the argument establishing the inequality $x_1^{2k+2} \leq x_1^{2k+1}$ relies on the fact that x_1^{2k+2} is the lowest best response to x_2^{2k+1}, and the fact that x_1^{2k+1} is the highest best response to x_2^{2k}. So, we obtain (a) and (b) by induction. ■

Proof. (of Lemma 3) Since $y_1^0 \geq x_1^0$, and $y_2^0 \leq x_2^0$, we have that $y_1^1 \leq x_1^1$ and $y_2^1 \geq x_2^1$ by (E) and because x_1^1 is the highest best response to x_2^0 and x_2^1 is the lowest best response to x_1^0. Similarly, if $y_1^{2k} \geq x_1^{2k}$ and $y_2^{2k} \leq x_2^{2k}$, then $y_1^{2k+1} \leq x_1^{2k+1}$ and $y_2^{2k+1} \geq x_2^{2k+1}$. This in turn implies that $y_1^{2k+2} \geq x_1^{2k+2}$ and $y_2^{2k+2} \leq x_2^{2k+2}$, by (E) and because x_1^{2k+2} is the lowest best response to x_2^{2k+1} and x_2^{2k+2} is the highest best response to x_1^{2k+1}. ■

Proof. (of Lemma 4) Part (a) and the first claim in part (b) are straightforward. To show the second claim of part (b), suppose that $\min\{x_1^K, x_1^{K+1}\} = x_1^K$. Then $\max\{x_1^K, x_1^{K+1}\} = x_1^{K+1}$, $\min\{x_2^K, x_2^{K+1}\} = x_2^{K+1}$, and $\max\{x_2^K, x_1^{K+1}\} = x_2^K$. Since x_2^{K+1} is a best response to x_1^K, $\min\{x_2^K, x_2^{K+1}\}$ is a best response to $\min\{x_1^K, x_1^{K+1}\}$. Similarly, since x_1^K is a best response to x_2^{K+1}, $\min\{x_1^K, x_1^{K+1}\}$ is a best response to $\min\{x_2^K, x_2^{K+1}\}$. So, $(\min\{x_1^K, x_1^{K+1}\}, \min\{x_2^K, x_2^{K+1}\})$ is a Nash equilibrium. Analogous arguments show that $(\max\{x_1^K, x_1^{K+1}\}, \max\{x_2^K, x_2^{K+1}\})$ is also a Nash equilibrium. ■