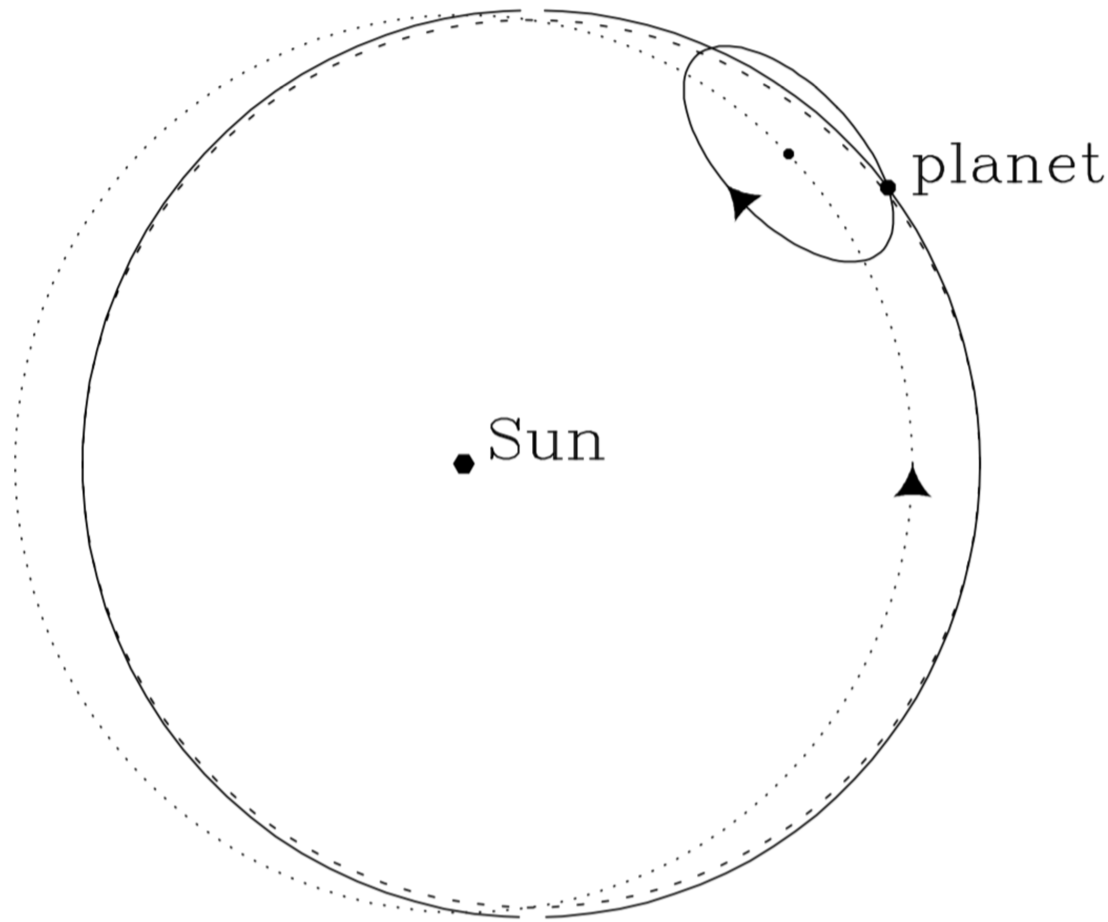
The epicyclic approximation for orbits in axisymmetric potentials

BT2, §3.2.3

Overview

In axisymmetric potentials, nearly circular orbits can be approximated as retrograde elliptical motion with **epicyclic frequency** κ on top of circular motion of a **guiding center** with **angular frequency** Ω

$$\kappa = \sqrt{d\Omega^2/d\ln R + 4\Omega^2}$$



Main steps of the epicyclic approximation derivation

- Write general EOMs in cylindrical coordinates
- Specialize to axisymmetric case and define effective potential $\boldsymbol{\Phi}_{\mathrm{eff}}$, such that *R, z* EOMs look like Cartesian EOMs in $\boldsymbol{\Phi}_{\mathrm{eff}}$
- Show that $\boldsymbol{\Phi}_{\mathrm{eff}}$ minimum (guiding center) follows a circular orbit around the center of the system
- Taylor expand $\boldsymbol{\Phi}_{\mathrm{eff}}$ around its minimum and show that motion in (x, y, z) coordinates in the frame of the guiding center can be approximated as simple harmonic oscillations with frequencies κ (in x, y) and ν (in z)
- Derive expression for κ in terms of the Ω profile of the system
- Obtain explicit solutions for $(x(t), y(t), z(t))$ elliptic motion

Gravitational dynamics in cylindrical coordinates

$$\mathbf{r} = R\hat{\mathbf{e}}_R + z\hat{\mathbf{e}}_z$$

$$\mathbf{v} = \dot{R}\hat{\mathbf{e}}_R + R\dot{\phi}\hat{\mathbf{e}}_\phi + \dot{z}\hat{\mathbf{e}}_z$$

$$\mathbf{a} = (\ddot{R} - R\dot{\phi}^2)\hat{\mathbf{e}}_R + (2\dot{R}\dot{\phi} + R\ddot{\phi})\hat{\mathbf{e}}_\phi + \ddot{z}\hat{\mathbf{e}}_z$$

$$\nabla\Phi = \frac{\partial\Phi}{\partial R}\hat{\mathbf{e}}_R + \frac{1}{R}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi + \frac{\partial\Phi}{\partial z}\hat{\mathbf{e}}_z$$

Equations of motion:

$$\mathbf{a} = -\nabla\Phi \quad \Rightarrow \quad \begin{cases} \ddot{R} - R\dot{\phi}^2 = -\dfrac{\partial\Phi}{\partial R} \\ 2\dot{R}\dot{\phi} + R\ddot{\phi} = \dfrac{1}{R}\dfrac{d(R^2\dot{\phi})}{dt} = -\dfrac{1}{R}\dfrac{\partial\Phi}{\partial\phi} \\ \ddot{z} = -\dfrac{\partial\Phi}{\partial z} \end{cases}$$

Axisymmetric case

$$\Phi = \Phi(R,\ z) \Rightarrow \frac{\partial \Phi}{\partial \phi} = 0 \qquad \Rightarrow \frac{d(R^2\dot{\phi})}{dt} = 0$$

$$\Rightarrow L_z = R^2\dot{\phi} = \text{const.}$$

z component of angular momentum

$$\Rightarrow \dot{\phi} = \frac{L_z}{R^2}$$

Reformulate *R*, *z* EOMs using this result:

$$\ddot{R} = -\frac{\partial \Phi}{\partial R} + \frac{L_z^2}{R^3} = -\frac{\partial \Phi_{\text{eff}}}{\partial R}$$

$$\ddot{z} = -\frac{\partial \Phi_{\text{eff}}}{\partial z}$$, where $\Phi_{\text{eff}} \equiv \Phi + \frac{L_z^2}{2R^2}$

I.e., motion in *R, z* modeled as oscillations in effective potential Φ_{eff}

Coordinates of the guiding center = $\Phi_{\rm eff}$ minimum

Coordinates $(R_{\rm g},\ \phi_{\rm g},\ z_{\rm g})$ of the $\Phi_{\rm eff}$ minimum satisfy:

$$0 = \frac{\partial \Phi_{\rm eff}}{\partial R} = \frac{\partial \Phi}{\partial R} - \frac{L_z^2}{R^3}$$

$$0 = \frac{\partial \Phi_{\rm eff}}{\partial z}$$

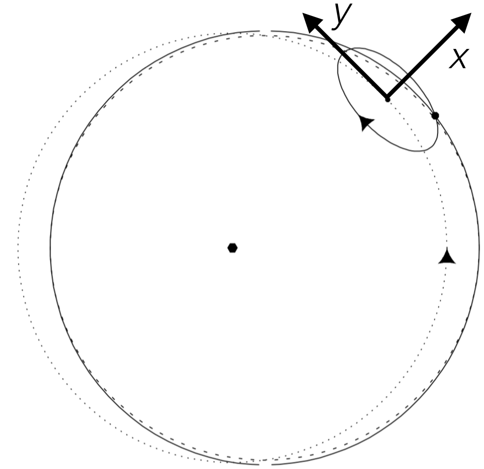
Assume that Φ is symmetric about *z*. Then the last equation is true everywhere in *z=0* plane and the first holds in that plane where

$$\left(\frac{\partial \Phi}{\partial R}\right)_{(R_{\rm g},\ 0)} = \frac{L_z^2}{R_{\rm g}^3} = R_{\rm g}\dot{\phi}_{\rm g}^2$$

centripetal acceleration of circular orbit of radius $R_{\rm g}$

I.e., *R, z* oscillate about circular orbit of radius $R_{\rm g}$ and angular momentum L_z

EOMs relative to guiding center



For small $x \equiv R - R_\mathrm{g}$,

0 (can add arbitrary const.)

0 ($(R_\mathrm{g}, 0)$ is $\mathbf{\Phi}_\mathrm{eff}$ minimum)

$$\Phi_\mathrm{eff} = \Phi_\mathrm{eff}(R_\mathrm{g}, 0) + \left.\frac{\partial \Phi_\mathrm{eff}}{\partial R}\right|_{(R_\mathrm{g},0)} x + \left.\frac{\partial \Phi_\mathrm{eff}}{\partial z}\right|_{(R_\mathrm{g},0)} z$$

0 ($\mathbf{\Phi}_\mathrm{eff}$ symmetric about $z=0$)

$$+\frac{1}{2}\left.\frac{\partial^2 \Phi_\mathrm{eff}}{\partial R^2}\right|_{(R_\mathrm{g},0)} x^2 + \frac{1}{2}\left.\frac{\partial^2 \Phi_\mathrm{eff}}{\partial z^2}\right|_{(R_\mathrm{g},0)} z^2 + \frac{1}{2}\left.\frac{\partial^2 \Phi_\mathrm{eff}}{\partial x \partial z}\right|_{(R_\mathrm{g},0)} xz + \ldots$$

In Cartesian *(x, z)* frame,

$$\mathbf{a} = -\nabla \Phi_\mathrm{eff}$$

$$\ddot{x} = -\frac{\partial \Phi_\mathrm{eff}}{\partial x} \approx -\left.\frac{\partial^2 \Phi_\mathrm{eff}}{\partial R^2}\right|_{(R_\mathrm{g},0)} x \equiv -\kappa^2 x$$

$$\ddot{z} = -\frac{\partial \Phi_\mathrm{eff}}{\partial z} \approx -\left.\frac{\partial^2 \Phi_\mathrm{eff}}{\partial z^2}\right|_{(R_\mathrm{g},0)} z \equiv -\nu^2 z$$

Harmonic oscillators with frequencies κ and ν

How to compute epicyclic frequency

Using definition $\Phi_{\rm eff} = \Phi + \frac{L_z^2}{2R^2}, \quad \kappa^2 = \left(\frac{\partial^2\Phi}{\partial R^2}\right)_{(R_{\rm g},0)} + \frac{3L_z}{R_{\rm g}^4}$

Circular angular frequency: $\Omega^2 = \frac{1}{R}\left(\frac{\partial\Phi}{\partial R}\right)_{(R_{\rm g},0)} = \frac{L_z^2}{R^4}$

(since for circular orbit

$\partial\Phi/\partial R = R\dot{\phi}^2;\ \Omega = \dot{\phi}$)

$$\Rightarrow \kappa^2(R_{\rm g}) = \Bigg(\underbrace{\Omega^2 + R\frac{d\Omega^2}{dR}}_{\partial^2\Phi/\partial R^2} + \underbrace{3\Omega^2}_{3L_z/R_{\rm g}^4}\Bigg)_{R_{\rm g}} = \left(R\frac{d\Omega^2}{dR} + 4\Omega^2\right)_{R_{\rm g}}$$

$$= \left(\frac{d\Omega^2}{d\ln R} + 4\Omega^2\right)_{R_{\rm g}}$$

Elliptic motion around guiding center

x: $\ddot{x} = -\kappa^2 x \quad \Rightarrow x(t) = X \cos(\kappa t + \alpha)$ for $X \geq 0$ and α constant

y: $\dot{\phi} = \frac{L_z}{R^2}$ L_z = const. in axisymmetric potential

$$= \frac{L_z}{R_g^2}\left(1 + \frac{x}{R_g}\right)^{-2}$$

$R = R_g + x = R_g(1 + x/R_g)$

$$\approx \frac{L_z}{R_g^2}\left(1 - \frac{2x}{R_g}\right)$$

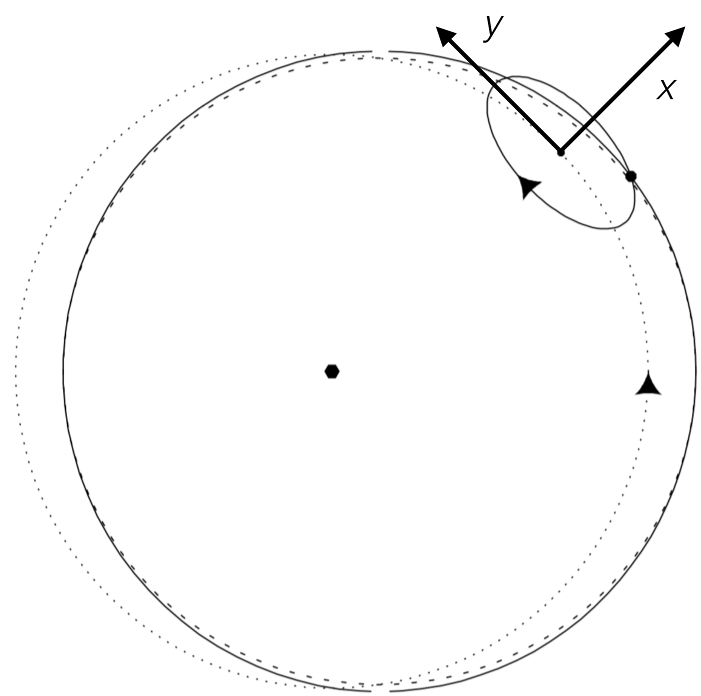
Taylor expansion

$$\Rightarrow \phi = \Omega_g t - \frac{2\Omega_g X}{R_g \kappa}\sin(\kappa t + \alpha) + \phi_0$$

integrating

$$y \approx R_g(\phi - \phi_g) = -\frac{2\Omega_g X}{\kappa}\sin(\kappa t + \alpha) \equiv -Y \sin(\kappa t + \alpha)$$

Summary of epicyclic motion



$$x = X\cos(\kappa t + \alpha)$$

$$y = -Y\sin(\kappa t + \alpha)$$

$$z = Z\cos(\nu t + \xi)$$ similarly as for *x*

Elliptic motion in *xy* plane with aspect ratio $\frac{X}{Y} = \frac{\kappa}{2\Omega_{\mathrm{g}}}$

Motion around the around the epicycle is opposite in sense to the rotation of of the guiding center. Conservation of $L_z = R^2\dot{\phi} \Rightarrow \dot{\phi}$ decreases when *R* increases