On the design of optimal auctions for road concessions: Firm selection, government payments, toll and capacity schedules with imperfect information

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A R T I C L E   I N F O

Article history:
Received 29 September 2020
Revised 1 March 2021
Accepted 1 March 2021

Keywords:
Public-private partnerships in transportation (PPPs)
P3s
Concession agreements
Capacity and tolling
Auctions
Mechanism design

A B S T R A C T

We consider firms with privately-known production efficiencies, captured in their cost structure, bidding for a road concession agreement with a government seeking to maximize the expected public welfare generated by the project. The setting is motivated by the increasing trend in road privatization around the world, and the need to design auctions leading to efficient outcomes: firm selection, government payments, toll and capacity schedules, which determine public welfare and firm profits.

In this paper, we characterize optimal direct revelation mechanisms for cases with and without restrictions on government payments. Because, in practice, it may be difficult or unappealing for firms to reveal their cost structure, we derive a scoring function that allows for the implementation of the optimal mechanism as a first-score auction where bids consist of toll, capacity, and government payment levels. The function accounts for distortions stemming from firms’ incentives to exploit their private information. We use the results to benchmark the performance of simple, but sub-optimal mechanisms: (i) predetermined toll and capacity bidding, (ii) auctions where the public welfare function is used to score bids, and (iii) a demand pricing mechanism aimed at maximizing patronage.

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1. Introduction

Private participation in the development of transportation infrastructure is ubiquitous, and is expected to grow further. In the US, for example, 32 states and Puerto Rico have enacted legislation enabling Public-Private Partnerships (Geddes and Wagner, 2013). In addition to development of new roads, i.e., design, construction, operations and management, governments can also franchise existing roads to private firms. For example, in 2005, the Skyway Corporation won a 99-year franchise for the Chicago Skyway, and became the first privatization of an existing road in the US (Enright, 2006). Private participation in transportation is also common in Europe. Albalate et al. (2009), for example, analyzed the level of private participation in tolled motorways in Europe. They found that 37% of roads (by length) are under concession agreements – most of them operated and maintained by private firms. The private sector plays a particularly important role in Southern Europe, where these trends are even more pronounced. There are also a number of (local) firms involved in the development of toll roads in China and elsewhere in Asia (Tan et al., 2010), where the implementation of P3s often takes the form

https://doi.org/10.1016/j.trb.2021.03.001
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of Build-Operate-Transfer agreements. In these agreements firms transfer the roads back to the government at the end of a fixed term (Yang and Meng, 2000). The Guangzhou–Shenzhen Super Highway Project, by a Hong Kong entrepreneur, is an emblematic example.

In addition to supplementing public funding, privatization of roads may have other advantages. Blom-Hansen (2003), for example, points out that that lack of competition for the public sector may reduce its incentives to perform efficiently. Small and Verhoef (2007) [p.201] mentions that the general public is often more accepting of paying tolls charged by private operators, as opposed to by public agencies.

At the same time, private provision of transportation infrastructure does raise significant concerns. This is due, in part, to the misalignment of incentives between firms seeking to maximize profit, and governments seeking to maximize social welfare. In particular, as explained in Small and Verhoef (2007) [p.193], both the public sector, if it decides to toll, and firms have the incentive to absorb congestion externalities. Firms, however, charge a mark-up that depends on the demand elasticity, and that reduces social welfare. These concerns are reflected in the literature, where papers in the last 2 decades have analyzed the effect of privatization on project outcomes, i.e., on tolling and capacity levels, and on social welfare in different settings: network topologies and ownership regimes. For example, the work of de Palma and Lindsey (2000) compares the social welfare of a 2-segment parallel road network, under 4 different ownership regimes: public-free, private-free, private-private and public-private. They find that the public-private duopoly can result in lower social welfare than the private-private one. Xiao et al. (2007) extend the analysis to the case of multiple parallel roads, each operated by a different firm, and show how additional competition affects the toll and capacity choices, and increases social welfare. Mun and Ahn (2008) study serial networks and find that the sum of the mark-ups set by independent, profit-maximizing operators exceeds the markup that would be charged by a single firm, i.e., a monopolist, operating the entire network. This means that decentralized, private operation of a pure serial network leads to reductions in both social welfare and in total operator profit! This phenomenon is referred to as double marginalization. In the same vein, Yang and Meng (2000, 2002) present a bi-level optimization framework to conduct the analysis on general networks with multiple OD pairs. They apply the framework to an example where they analyze the concession of a two-directional link in an inter-city expressway network in the Pearl River Delta Region of South China. They split the toll-capacity plane into 4 regions depending on whether the concession is profitable or not, and whether or not it increases social welfare vis-à-vis the network without the link.

In addition to identifying market structures that might be more amenable to privatization, a number of approaches have been studied in the literature to improve the welfare generated by road concessions. Among them, we note contract provisions restricting toll rates, toll revenues, capacity, rates of return (Tsai and Chu, 2003; Tan et al., 2010), government payments/subsidies or tax incentives (Zhang and Durango-Cohen, 2012), and, most relevant to the work herein, the use of auctions to award concessions.

At a high level, 2 types of approaches have appeared in the literature in the analysis of auctions for road franchising: the first, largely inspired by practice, where firms bid for a road concession where the quality variables, i.e., toll and capacity levels, are predetermined by the government; and the second, where quality variables are selected by firms as part of the bidding strategy.

In the context of auctions with predetermined quality variables, minimum-cost auctions, where bidders submit cost estimates and the bidder with lowest cost wins, are the simplest and most widely-studied and used framework. Extensions include A+B bidding, where the cost associated with the time to complete the project, B, is added to the project costs, A, to obtain the total cost of the bid (Herbsman, 1995). El-Rays and Kandil (2005) further extend the A+B cost measure to include a quality measure C. Other approaches considered in the literature include adding revenue metrics to the cost measure. Engel et al. (1997), for instance, introduce the least present value of revenue (LPVR) auction, where bids are for the present value of toll revenue that they propose to collect over the duration of the franchise agreement. The firm that bids the LPVR wins the franchise. From the government’s perspective, LPVR is appealing because firms compete. From a firm’s perspective, LPVR is attractive because it mitigates risks associated with demand uncertainty, and other factors, because franchise agreements extend until the specified LPVR is realized. Nombela and de Rus (2004) extend the LPVR to include costs. They refer to the criterion as the least present value of net revenue (LPVNR), where franchises are awarded on the basis of profit/income.

The simplicity and flexibility of these auctions explain their appeal. The papers above analyze their performance in settings involving uncertainty, government guarantees, renegotiation opportunities, etc. Disadvantages that serve as motivation for the work herein include:

1. The design burden, i.e., determining (optimal) toll and capacity levels, falls on the government. This is unappealing because governments may lack the technical expertise, or importantly, the information that is necessary. In particular, welfare maximizing toll and capacity levels depend on a firm’s production efficiency as reflected in its cost structure, which is unlikely to be known to the government in advance of the auction.
2. The selection criteria are not aligned with the government’s objective to maximize social welfare. In particular, firms, even those with a competitive advantage due to their high production efficiency, do not have incentive to build roads that exceed the minimum requirements because costs increase, thereby reducing the probability of winning.

Auctions with predetermined quality variables are discussed further in Section 3. In terms of auctions where firms determine the levels of quality variables as part of their bidding strategies, Verhoef (2007); Ubbels and Verhoef (2008) are seminal studies in the context of road concessions. Their focus is to understand how the criterion used to select the winning
bid, i.e., the scoring function, impacts the welfare generated by the project in different settings, i.e., roads in isolation, roads interacting with an untoll complement or substitute, and networks in general. They explain that the choice of criterion is critical in designing auctions where the bidders strategies are contingent on private information. Because the objective is to assess and compare scoring functions, rather than developing a model that captures the information structure, they assume that the auctions are perfectly competitive, which allows them to determine the project outcomes by optimizing the aforementioned criteria subject to the constraint that firms earn normal/zero profits. The constraint ensures that firms have incentive to participate in the auction. Among the interesting results, they find that using demand/patronage maximization as a criterion maximizes social welfare, i.e., it yields the first-best solution for roads in isolation, and the second-best solution for roads in networks when spillovers exist.

In this paper, we relax the assumption of perfectly competitive auctions, which is predicated on having a large number of homogeneous bidders, no barriers to firm entry/exit, complete and perfect information, etc. Indeed, studies, such as Jofre-Bonet and Pesendorfer (2000), indicate that, in practice, auctions for road concessions often attract small numbers of firms that may differ greatly in terms of their capabilities and production efficiencies. Verhoef (2007), among others, explains that these differences reflect privately-held business practices, expertise and experience, etc. Following the bargaining model presented in Shi et al. (2016), we assume that these differences manifest themselves in the firms’ variable construction costs, which are represented as iid random variables drawn from a probability distribution function that is common knowledge to the firms and the government.\footnote{The assumptions of stationary traffic, and of homogeneous users follow Verhoef (2007); Ubbels and Verhoef (2008).} We apply the framework of Myerson (1981) to design optimal road concession auctions yielding Nash Equilibrium bids, consisting of toll and capacity strategies, that maximize a project’s expected social welfare. The framework relies on the Revelation Principle where, given the opponents’ strategies, firms find it in their interest to reveal their private information, i.e., their variable costs, as part of the bids (Myerson, 1979). In turn, this allows governments to allocate projects to the most efficient firms. We characterize optimal bidding strategies and observe that firms have incentive to exploit their private information by proposing roads with lower capacity and higher tolls than those obtained under perfect information or in the case of perfectly competitive auctions. The assumption of complete information and the Revelation Principle mean that, in turn, the government can extract profits that may stem from this distortion. Project outcomes depend on the realization of the bidders’ variable costs. The outcomes approach those obtained under perfectly competitive auctions when there are a large number of bidders.

Following Che (1993), we formulate a scoring function that can be used to implement the optimal direct mechanism as a first-score auction. The scoring function consists of 2 parts: the social welfare, and a term that controls for bidders incentives to exploit their private information. We also present performance bounds for 2 simple, but sub-optimal alternatives: a naïve scoring auction where the scoring function corresponds to the government’s welfare maximization objective, but does not account for distortions; and, inspired by the patronage maximization auction of Verhoef (2007), we present a demand pricing mechanism where firms select bundles consisting of a demand level and an associated government payment. The mechanism is motivated by the complexity of evaluating scoring functions. Numerical examples show that the demand pricing mechanism performs almost as well as the optimal mechanism because it partially accounts for distortions.

To conclude this section, we highlight key contributions of our work:

- This paper advances the literature on auctions for road concessions where quality variables, i.e., tolls and capacity, are determined as part of the bidding process by extending the analysis to the case of imperfect information, where government’s are not privy to firms’ cost structures. The benchmark papers of Verhoef (2007); Ubbels and Verhoef (2008) mention that information asymmetries are inherent in interactions among governments and firms, but, to the best of our knowledge, the case has not been studied elsewhere in the transportation (economics) literature. We do acknowledge the bargaining model in Shi et al. (2016) from which we borrow the information structure used in our analysis.

- In addition to characterizing optimal mechanisms, the framework of Myerson (1981) provides an established approach to study related issues/problems in the process of awarding road concessions through auctions, e.g., regulatory concerns (Laffont and Tirole, 1986; McAfee and McMillan, 1987; Riordan and Sappington, 1987), design of combinatorial auctions to award (multiple) concessions within transportation networks (Cramton et al., 2010), to account for systematic and random demand fluctuations (Pavan et al., 2014; Strausz, 2006), etc. Because the work constitutes one of the first applications of the mechanism design in transportation, the analysis can be used as a template and to provide insight into the design and performance of auctions for concessions in synergistic settings, such as public transit lines, port and other facilities, parking services/infrastructure, etc., where PPPs are increasingly common.

- Following Che (1993) who formalizes the fundamentals of scoring auctions where bids consist of multidimensional arrays of product characteristics, we describe implementation of the optimal direct revelation mechanism as a first-score auction. The framework’s flexibility explains its broad appeal and use in applications ranging from allocation of airport time slots (Rassenti et al., 1982), to procurement of weapons systems (Che, 1993), to online auctions for goods and services (Hartline, 2020). We also highlight development of the demand pricing mechanism, presented herein, as a contribution with potentially significant implications, as it provides an appealing, practical and near-optimal framework to design auctions for road concessions.
The remainder of the paper is organized as follows. In Section 2, we review the problem of setting social welfare maximizing toll and capacity levels with complete and perfect information. We also consider variations intended to ensure firm participation. We begin Section 3 by introducing assumptions to structure the analysis of road concession auctions with imperfect information. We analyze 2 benchmark situations. First, we consider auctions with predetermined toll and capacity levels. We then apply the framework of Myerson (1981) to the design of optimal auctions for cases with and without restrictions on government payments. Implementation of the optimal direct revelation mechanism as a scoring auction is discussed in Section 4, where we also consider simple, but sub-optimal alternatives, including a generalization of the patronage maximization auction of Verhoef (2007). Numerical examples to compare the various mechanisms are presented in Section 5. A summary and conclusion appears in Section 6.

2. Toll and capacity choices with complete and perfect information

We consider a government wishing to franchise a road construction and operation project. Road construction costs are associated with materials, personnel, equipment, and financing. We follow Small and Verhoef (2007) [p.106, Eq.3.4.1] and the references therein, where it is explained that they can be represented by the sum of a fixed cost term, and a linear variable cost term that is a function of road capacity. Empirical evidence, see e.g., Levinson and Gillen (1998), suggests that (fixed and variable) costs associated with equipment and material procurement do not vary widely across firms, but that there are significant differences across firms in variable costs associated with labor and financing costs. As shown in (1), we consider a specification where the fixed costs are homogeneous across firms, and following Shi et al. (2016), the variable cost per-unit of capacity is firm-dependent.

\[ C_i^f(K) = c_0 + c_i \cdot K \]  

(1)

where \( c_0 > 0 \) is the fixed cost, \( c_i \geq 0, i = 1, \ldots, I \) is the \( i \)th firm’s variable cost. \( K \) denotes road capacity, and is measured in number of vehicles per hour.\(^2\) Following Verhoef (2007), the social welfare or aggregate surplus, and operating profit associated with awarding a concession to firm \( i \) to build a road of capacity \( K \) are given as follows:

\[ SW_i(N, K) = \int_{n=0}^{n=N} P(n)dn - N \cdot C(N, K) - C_i^f(K) \]  

(2)

\[ \pi_i(\tau, N, K) = N \cdot \tau - C_i^f(K) \]  

(3)

The variables \( \tau \) and \( N \), respectively, represent the toll charge and the number of users, i.e., the demand/traffic, traveling through the road segment. \( P(n) \) is the inverse demand function, and corresponds to \( n \)th user’s willingness to pay for travel, i.e., the \( n \)th user’s utility of travel. \( C(N, K) \) is the average travel cost per user. As is done elsewhere, we refer to \( C(N, K) \) as the per-user average congestion cost function. It converts travel time to travel costs.

We make the following assumptions for model tractability:

**Assumption 1.** Demand and cost functions: First and second order conditions

1. The inverse demand function, \( P(n) \), is twice differentiable, decreasing, and concave, i.e., \( P'(n) = \frac{\partial P(n)}{\partial n} < 0 \) and \( P''(n) = \frac{\partial^2 P(n)}{\partial n^2} \leq 0 \).

2. The congestion cost function is given as \( C(N, K) = g(\frac{N}{K}) \), which is homogeneous of degree 0. Further, we let \( \mu = \frac{N}{K} \) and assume that \( g(\mu) \) is twice differentiable, increasing, and convex, i.e., \( g'(\mu) = \frac{\partial g(\mu)}{\partial \mu} > 0 \) and \( g''(\mu) = \frac{\partial^2 g(\mu)}{\partial \mu^2} \geq 0 \).

\( \mu \) is referred to as the *volume-to-capacity* or *v/c* ratio. Here, we assume that the relationship among operator’s decision variables, \( \tau \) and \( K \), and the demand, \( N \), is given by the user *equilibrium condition*, described in Assumption 2 below.

**Assumption 2.** User Equilibrium

The \( N \)th user’s willingness to pay equals the total generalized travel cost; that is,

\[ P(N) = \tau + C(N, K) \]

which allows us to write the demand as a function of the decision variables as \( N(\tau, K) \).

\(^2\) Small and Verhoef (2007) [p.105] defines \( C_i^f(K) \) as the capital cost of building roads. To simplify the notation, we define \( C_i(K) \) as the equivalent uniform costs per period, which eliminates the product of the capital recovery factor with Small and Verhoef’s \( C_i^f(K) \). Because the concession agreement is for a given road, the effect of road length, terrain, etc. is implicitly included in the parameters \( c_0 \) and \( c_i \). The spirit of the model is to capture all costs borne by an operator in \( C_i(K) \). Thus, the specification may be inadequate/inaccurate in cases where there are large cost components that depend on variables other than capacity. For example, in addition to capacity, thickness/ durability has been shown to have a significant effect on construction costs, and maintenance costs have been represented as functions of capacity, thickness, and (the induced) demand/traffic (Small et al. 1989 [p.22–36]). With limitations acknowledged, it is important to note that the cost structure adopted herein, specified as a function of 1 variable –capacity–, is identical to the relevant benchmarks Verhoef (2007); Ubbels and Verhoef (2008); Shi et al. (2016), and in Small and Verhoef (2007) [p.106, Eq.3.4.1].
The above specification and assumptions lead to the following social welfare maximization problem, the First-Best, $FB$, problem, reflecting the government’s ex-post objective subsequent to awarding the road concession to firm $i$:

$$
FB \quad \max_{\tau,K} SW_i(N,K) = \int_0^N P(n)dn - N \cdot C(N,K) - \mathcal{C}_i^*(K)
$$

subject to: \hspace{10pt} P(N) = \tau + C(N,K)

From the first-order conditions, we have

$$
\begin{align*}
\tau^* &= N - C_N \\
K^* &= -N \cdot C_K |_{K^*} = c_i
\end{align*}
$$

(4)

where $C_N \equiv \frac{\partial C(N,K)}{\partial N}$ and $C_K \equiv \frac{\partial C(N,K)}{\partial K}$. The optimal toll, $\tau^*$, internalizes the congestion externality, $N \cdot C_N$. This is the well-known result of Pigou (1932). We also note that the optimal capacity depends on a firm’s efficiency, captured in $c_i$. Allowing for the congestion cost specification in Assumption 1.1,

$$
\mu^* : g'(\mu) \cdot \mu^2 \big|_{\mu=c_i} = c_i
$$

(5)

Eq. (5) implies that the optimal $v/c$ ratio only depends on $g(\cdot)$ and $c_i$. Once the optimal $v/c$ ratio, $\mu^*$, is determined, the optimal values of the decision variables are given as follows: $^3$

$$
\begin{align*}
\tau^* &= \mu^* \cdot g'(\mu^*) = \frac{c_i}{\mu^*} \\
N^* &= P^{-1}(\tau^* + g(\mu^*)) \\
K^* &= \frac{N^*}{\mu^*}
\end{align*}
$$

Letting $\mu^*(c_i)$ represent the optimal solution for variable cost $c_i$, it follows from Assumption 1.2 that $\mu^*(c_i)$ is increasing in $c_i$, i.e., $\frac{\partial \mu^*(c_i)}{\partial c_i} > 0$. This means higher variable costs lead to more congestion. Similarly, optimal tolls, capacities, and the induced demand are, respectively, increasing, decreasing, and decreasing in $c_i$. Again, letting the variables be defined as functions of $c_i$, $\frac{d\tau^*(c_i)}{dc_i} > 0$, $\frac{dN^*(c_i)}{dc_i} < 0$, and $\frac{dK^*(c_i)}{dc_i} < 0$. It follows that social welfare is decreasing in $c_i$.

In terms of the firm’s operating profit, Eq. (3), we note that $\pi_i(\tau^*, N^*, K^*) = -c_0 < 0$. That is, implementation of the optimal solution to $FB$ does not allow firms to recover fixed costs, $c_0$, and thus, may hinder participation. Solutions analyzed in the literature include (i) government payments to compensate firms for the loss (or to ensure a minimum profit); and (ii) imposing the constraint that toll revenues cover costs. We elaborate on both approaches below.

In terms of the first approach, we consider a variation where a government interested in maximizing public welfare, $PW(\cdot)$, subject to the constraint that the firm’s total profit is nonnegative.

$$
PW \quad \max_{\tau,K} PW(\tau, N, K, M) = CS(\tau, N, K) - M
$$

$$
= \int_0^N P(n)dn - N \cdot C(N,K) - N \cdot \tau - M
$$

subject to: \hspace{10pt} P(N) = \tau + C(N,K)

$$
\pi_i(\tau, N, K) + M \geq 0
$$

where $CS(\cdot)$ is the user/consumer surplus, given by the difference between total travel benefits and costs, $M$ is a government payment to the firm, and the left-hand-side of the second constraint represents the firm’s total profit. $^4$ $PW(\cdot)$ corresponds to the welfare realized by the users and the government, i.e., it is the social welfare that is left when the firm’s total profit, $\pi_i(\cdot) + M$, is subtracted. Mathematically, $SW_i(\cdot) = CS(\cdot) - M + N\tau - \mathcal{C}_i^*(\cdot) + M = PW(\cdot) + \pi_i(\cdot) + M$. The first equality shows that the social welfare, $SW_i(\cdot)$, is independent of the payment, $M$, and of the toll revenues, $N\tau$, because they are transfers between the parties. Because the profitability constraint is binding at optimality, the solutions to $FB$ and $PW$ are equivalent with $M^* = -\pi_i(\tau^*, N^*, K^*) = c_0 > 0$. That is, in the case of complete and perfect information, (i) the solution to $PW$ maximizes social welfare, and (ii) yields a strict preference for the minimum payment that ensures participation. The latter is appealing because solutions with payments above the minimum generate the same social welfare by increasing firm profits at the expense of public welfare. The 2 observations explain why public welfare maximization is used when there are government payments. In addition to ensuring participation, in the case of imperfect information, government payments are used as a tool to control firms’ incentives to exploit information asymmetries.

$^3$ Assumption 1 and $P(0) > \frac{c_i}{\mu^*} g(\mu^*)$ ensure that $N^* > 0$, i.e., that the solution is interior, and that $\tau^*, N^*, K^*$ is the unique optimal solution to $FB$.

$^4$ Without loss of generality, the right-hand-side of the constraint can be used to impose a requirement that the profit exceed a given threshold — when there are opportunity costs associated with participation.
Introducing direct payments ensures viability from firm \( i \)'s standpoint, but may be unappealing when governments have limited resources. This motivates the second approach, where tolls, capacities, and the induced demand solve the optimization problem introduced by Ramsey (1927), \( \text{RP} \).

\[
\text{RP} \quad \max_{\tau, K} SW_i(N, K) = \int_0^N P(n)dn - N \cdot C(N, K) - C_i^\tau(K)
\]

subject to:

\[
P(N) = \tau + C(N, K)
\]

\[
\pi_i(\tau, N, K) \geq 0
\]

As before, the second constraint can be used to impose a minimum profit requirement. Implementation of solutions to \( \text{FB/PW} \) or \( \text{RP} \) requires complete and perfect information. Hereafter, we assume the realizations of \( c \) are unknown to the government, which complicates the process of awarding a concession (to the most efficient firm), and may lead to outcomes that differ from those prescribed by \( \text{FB/PW} \) or \( \text{RP} \).

3. Analysis of road concessions via auctions

In this section, we analyze auctions designed to award road concessions in cases where governments are not privy to firms’ cost structures, i.e., where there is imperfect information. To model the situation, as is done in the auctions literature (Vickrey, 1961; Myerson, 1981), and in the road franchising model presented in Shi et al. (2016), we assume:

**Assumption 3.** Variable costs per-unit of capacity

The variable costs per-unit of capacity for each firm are iid random variables drawn from the probability density function, \( f(\cdot) \), with finite support, i.e., \( c_i \in C = [c_{\text{min}}, c_{\text{max}}], \quad i = 1, \ldots, I \). Letting \( c \equiv [c_1, \ldots, c_I] \) denote the collection of \( I \) firms’ variable costs, we can write the joint probability density function of \( c \) as

\[
\phi(c) = \prod_{i=1}^I f(c_i)
\]

Thus, the fixed costs and the density function are common knowledge. The realization of \( c_i \) is firm \( i \)'s private information.

We consider 2 benchmarks. First, auctions where the government predetermines toll and capacity levels, and where the bid with the minimum payment, \( M \), wins the franchise. We then apply the framework of Myerson (1981) for the design of optimal auctions, where bidders reveal their private information, and select toll and capacity levels that maximize the project’s expected social welfare. In turn, governments are able to allocate projects to the most efficient firm, and to extract profits that may stem from distortions. We present and analyze optimal direct revelation mechanisms for cases with and without government payments. The former is analogous to \( \text{PW} \), and the latter to \( \text{RP} \).

3.1. Auctions with predetermined toll and capacity levels

In this section, we consider auctions where governments set requirements for toll and capacity levels. We assume that welfare is optimized based on an estimate of the variable cost, \( \hat{c} \), which is used to set \( \tau^*(\hat{c}) \) and \( K^*(\hat{c}) \).\(^5\) The government payment, \( M \), is the key bid element. Given a predetermined toll and capacity parametrized by \( \hat{c} \), the operating profit of an arbitrary firm with variable cost, \( c_i \):

\[
\pi(c_i; \tau^*(\hat{c}), N^*(\hat{c}), K^*(\hat{c})) = N^*(\hat{c})\tau^*(\hat{c}) - C_i^\tau(K^*(\hat{c})) = N^*(\hat{c})\tau^*(\hat{c}) - (c_0 + \hat{c} \cdot K^*(\hat{c})) + (c_0 + \hat{c} \cdot K^*(\hat{c})) - (c_0 + c_i \cdot K^*(\hat{c})) = (\hat{c} - c_i)K^*(\hat{c}) - c_0
\]

as defined earlier, \( N^*(\hat{c}) \) is the demand induced by \( \text{FB} \) maximizing toll and capacity levels for variable cost \( \hat{c} \). \( N^*(\hat{c}) \equiv N^*(\tau^*(\hat{c}), K^*(\hat{c})) \). The firm’s revenues depend on \( \hat{c} \) but its costs are based on its actual variable cost, \( c_i \). We observe that \( \pi(c_i; \tau^*(\hat{c}), N^*(\hat{c}), K^*(\hat{c})) \) decreases linearly with \( c_i \). Having specified the probability density function for firm \( i \)'s variable costs, \( f(c_i) \), for a given \( \hat{c} \), the corresponding probability density function of firm \( i \)'s profit, \( h(\hat{c}; y) \), is given by

\[
h(\hat{c}; y) = f(\pi^{-1}(y; \tau(\hat{c}), N(\hat{c}), K(\hat{c}))), \quad y \in [\pi^\text{min} = \pi(e_{\text{max}}; \tau(\hat{c}), N(\hat{c}), K(\hat{c})), \pi^\text{max} = \pi(e_{\text{min}}; \tau(\hat{c}), N(\hat{c}), K(\hat{c}))]
\]

where \( \pi^{-1}(\cdot) \) is the profit function’s inverse. This shows that considering a firm’s operating profit as its private information is equivalent to its variable cost.

In minimum cost auctions with predetermined toll-capacity levels, bidder \( i \) submits its requested government payment, \( M_i \). Thus, optimal bidding strategies maximize \( \{\pi(c_i; \tau^*(\hat{c}), N^*(\hat{c}), K^*(\hat{c})) + M_i\} \cdot P(\text{winning with } M_i) \). In LPVR (Enright, 2006)

\(^5\) \( \hat{c} \) may, for example, be obtained by calculating the expected variable cost, \( E[c_i] \).
or LPVNR (de Rus and Nombela, 2000) auctions, firms bid on the revenue. Thus, bidding strategies maximize
\[
\{ M_i - C_i(\bar{c}) \} P(\text{winning with } M_i) = \{ M_i + \pi(c_i) M_i \} \tau^*(\bar{c}), N^*(\bar{c}), K^*(\bar{c}) \cdot (N^*(\bar{c}) - N^*(\bar{c})) P(\text{winning with } M_i)
\]
Both types of auctions lead to the same bids because \( N^*(\bar{c}) \cdot \tau^*(\bar{c}) \) is constant. The outcomes of predetermined toll-capacity bidding are presented below.

**Proposition 1.** Outcomes of predetermined toll-capacity bidding

1. Bidder's interim surplus, i.e., the conditional expected profit, is
\[
\int_{-\infty}^{\mu_i} H^{-1}(y) \, dy = K^*(\bar{c}) \int_{c_i}^{\mu_i} F_{\mu_i}(x) \, dx
\]
where \( H(\cdot) \) and \( F(\cdot) \) respectively, represent the cumulative distribution functions corresponding to density functions \( h(\cdot) \) and \( f(\cdot) \). \( \mu_i = \pi(c_i) \tau^*(\bar{c}), N^*(\bar{c}), K^*(\bar{c}) \) denotes firm \( i \)'s ex-post profit.

2. The ex-ante public welfare is
\[
E_w \left[ CS(\tau^*(\bar{c}), N^*(\bar{c}), K^*(\bar{c})) + \pi(w; \tau^*(\bar{c}), N^*(\bar{c}), K^*(\bar{c})) - \frac{F(w)}{F(\mu)} \cdot K^*(\bar{c}) \right]
\]
where \( E_w \) denotes the expectation over the winning firm's variable cost, \( w \).

**Proof.** Eq. (8) follows the result of Riley and Samuelson (1981) for first-price auctions. A derivation of (9) is in Appendix A. The sum of the first two terms inside the expectation of (9) is the ex-post social welfare when the winning firm has a variable cost of \( w \). The third term is a distortion related to the distribution, which will be extensively discussed in the following sections. Toll and capacity levels depend on \( \bar{c} \) rather than on the winner's variable cost. □

3.2. Optimal direct revelation mechanisms

In the context of road concessions, a mechanism is a framework/game whereby firms simultaneously and confidentially submit bids that depend on their private information. Governments, in turn, decide how to award/allocate the concession, and how to compensate each of the firms for their participation. The allocation and payment rules depend on the bids that are received. For a given mechanism, consisting of bid requirements, allocation and compensation rules, the intent is to predict the outcome, or at least to characterize outcomes and the underlying bidding strategies as being Nash Equilibria, i.e., strategies where no firm has incentive to deviate. A government is, of course, interested in designing mechanisms that are optimal with respect to its objective. In particular, given the joint probability density function for the set of variable costs for all firms, \( \phi(\bar{c}) \), a government is interested in designing mechanisms that maximize the expected public/social welfare.

The key insight that enables analysis of the ambiguous and complex mechanism design problem is that there is no loss of generality in considering only direct revelation mechanisms, where bids consist of firms’ private information (Myerson, 1981). This insight stems from the Revelation Principle of Myerson (1979), which states that each (Bayesian) Nash Equilibrium outcome in any feasible mechanism can be obtained as a (Bayesian) Nash Equilibrium in a direct mechanism with truthful reporting. In the context of road concessions, the direct revelation mechanism, \( (Q, M, T, L) \), can be described as the process below:

1. In advance of the auction, the government makes the toll and capacity schedules, as well as compensation and allocation rules publicly-available to all potential bidders. Mathematically, the respective functions are defined as \( T : \mathcal{C} \mapsto \mathbb{R}^+ \), \( L : \mathcal{C} \mapsto \mathbb{R}^+ \), \( M \equiv \{ M_1, \ldots, M_l \} \) where \( M_i : \mathcal{C}^l \mapsto \mathbb{R}, i = 1, \ldots, l \), and \( Q \equiv \{ q_1, \ldots, q_l \} \) where \( q_i : \mathcal{C}^l \mapsto \{ 0, 1 \} \) and \( \sum_{i=1}^{l} q_i(\bar{c}) = 1 \). \( \mathcal{C}^l \). \( ^8 \)

2. Each private firm reports its variable cost, which the government collects in \( \bar{c}. \) A bidder reporting \( c_i \) commits to build a road of capacity \( L(c_i) \) and to set the toll at \( T(c_i) \), if it wins the franchise.

3. The government uses \( Q \) to award the franchise. Because \( Q \) specifies exactly 1 winner for each set of variable costs in \( \mathcal{C} \), a tie-breaking rule may be necessary for cases where multiple firms report the same variable costs. Government compensation to the firms is captured in \( M(\bar{c}). \) The amount received by firm \( i, M_i(\bar{c}), \) could be negative, which means the firm pays the government.

In the remainder of the section, we formulate and solve 2 instances of the problem of designing direct revelation mechanisms that maximize the project’s expected public/social welfare. The design problem involves full characterization of the functions \( (Q, M, T, L) \), i.e., functional forms and parameters, constituting the game’s/auction’s rules. We begin, however, by discussing 2 conditions that must be satisfied by direct revelation mechanism: (i) a direct revelation mechanism must induce each bidder to report their true variable costs; and (ii) the mechanism needs to ensure that bidders have sufficient

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6 The difference between least present value of revenue (LPVR) and least present value of net revenue (LPVNR) auctions is that the in the latter, the net revenue excludes costs.
7 All parties in the game are assumed to be self-interested, rational, and non-cooperative.
8 In terms of notation, \( \mathbb{R}^+ \) represents the set of nonnegative real numbers, \( \mathcal{C}^l \) is the Cartesian product of set \( \mathcal{C} \) with \( \mathcal{C}^{l-1} \) where \( \mathcal{C}^1 = \mathcal{C} \).
incentive to participate. The first condition is referred to as Incentive Compatibility (IC), and the second condition is referred to as Individual Rationality (IR). Relevant versions of the 2 conditions are presented below.

**Definition 1.** Bayesian Incentive Compatibility

For \((Q, M, T, L)\), bidder \(i\)'s expected utility of reporting \(\bar{c}_i\), \(U_i(\cdot)\), is defined as\(^9\)

\[
U_i(\bar{c}_i, c_i) = E_{c_{-i}}[q_i(\bar{c}_i, c_{-i}) \cdot \pi(c_i; T(\bar{c}_i), N(\bar{c}_i), L(\bar{c}_i)) + M_i(\bar{c}_i, c_{-i})]
\]  

(10)

The expectation is taken over the variable costs of the \(I - 1\) firms other than firm \(i\), denoted \(c_{-i}\). Further, \((Q, M, T, L)\) is said to be Bayesian incentive compatible if and only if

\[
V_i(c_i) \equiv U_i(c_i, c_i) \geq U_i(\bar{c}_i, c_i) \quad \bar{c}_i \in C, i = 1, \ldots, I.
\]  

(11)

where \(V_i(c_i)\) is the expected utility of the truth-telling strategy.

Eq. (10) captures the fact that the reported variable cost impacts a firm's chance of winning, toll and capacity levels, as well as the payment it receives. It does not, however, impact its cost structure. The above condition guarantees that, at the Bayesian Nash Equilibrium, each firm \(i\)'s strategy is to report its true variable cost. The Generalized Monotonicity Condition, presented as **Proposition 2** below, provides a tractable form of IC condition.

**Definition 2.** Noting that **Assumption 1**, guarantees \(N(\cdot)\) and \(\pi(\cdot)\) are differentiable, for mechanism \((Q, M, T, L)\), let

1. \(\pi(c_i; T(\bar{c}_i), N(\bar{c}_i), L(\bar{c}_i))\) denote the (operating) profit of a winning firm with variable cost \(c_i\) claiming \(\bar{c}_i\).
2. \(G_i : C^2 \rightarrow \mathbb{R}\) as,

\[
G_i(\bar{c}_i, c_i) = E_{c_{-i}}[q_i(\bar{c}_i, c_{-i}) \cdot \pi'(c_i; T(\bar{c}_i), N(\bar{c}_i), L(\bar{c}_i))]
\]

where \(\pi'(\cdot)\) is the partial derivative of profit function with respect to the true variable cost, \(\pi'(c_i) \equiv \frac{\partial \pi(c_i)}{\partial c_i}\). Thus, \(G_i(\bar{c}_i, c_i)\) is an auxiliary function corresponding to the partial derivative of \(U_i(\bar{c}_i, c_i)\) with respect to type \(c_i\).

**Proposition 2.** **Generalized Monotonicity Condition**

\((Q, M, T, L)\) satisfies Bayesian IC if and only if

1. \(S_i(c_i) \equiv G_i(c_i, c_i)\), i.e., \(S_i(c_i)\) is the value of \(G_i(\cdot)\) when firm \(i\) claims \(c_i\), i.e., \(\bar{c}_i = c_i\).

\[
V_i(c_i) = V_i(\bar{c}_i) + \int_{\bar{c}_i}^{c_i} S_i(x) \, dx \quad \forall c_i' \in C, i = 1, \ldots, I
\]

and

\[
G_i(c_i, x) - G_i(x, x) \geq 0. \quad \forall c_i' \in C, i = 1, \ldots, I
\]

The proof appears in **Appendix A**. The first part of the condition is a definition, and the second part enforces the IC requirement, i.e., firm \(i\) is never penalized for revealing its type, \(c_i\). As is shown in Myerson (1981); Jehiel et al. (1999) in the context of the private value auctions

\[
\int_{\bar{c}_i}^{c_i} (G_i(c_i, x) - G_i(x, x)) \, dx \geq 0 \iff \int_{\bar{c}_i}^{c_i} (P_i(c_i) - P_i(x)) \, dx \geq 0
\]

(12)

where \(P_i(c_i) \equiv E_{c_{-i}}[q_i(\bar{c}_i)] = E_{c_{-i}}[q_i(c_i, c_{-i})]\), \(c_i \in C, i = 1, \ldots, I\), and corresponds to firm \(i\)'s interim probability of winning, i.e., the probability of winning given the firm's own type, \(c_i\).\(^{10}\) Also, \(P_i(c_i)\) is monotonically-decreasing in \(c_i\).

Direct revelation mechanisms must also ensure firm participation. With complete and perfect information, each bidder must earn a zero profit. For the case of imperfect information, **Definition 3** presents two versions of the IR condition.

**Definition 3.** Individual Rationality

1. **Interim IR**: At the interim stage, i.e., given \(c_i \in C, i = 1, \ldots, I\) and assuming IC, each bidder's conditional expected utility of participating is higher than the utility of the outside option.

\[
V_i(c_i) \geq 0 \quad i = 1, \ldots, I
\]

(13)

2. **Ex-post IR**: For \((Q, M, T, L)\), given \(c_i \in C\), and \(c_{-i} \in C^{I-1}\) and assuming IC, firm \(i\)'s ex-post utility is

\[
u_i(c_i, c_{-i}) = q_i(c_i, c_{-i}) \cdot \pi(c_i; T(c_i), N(c_i), L(c_i)) + M_i(c_i, c_{-i})
\]

(14)

Ex-post IR holds if and only if,

\[
u_i(c_i, c_{-i}) \geq 0 \quad i = 1, \ldots, I
\]

(15)

---

\(^9\) Abusing the notation, \(N(\bar{c}_i)\) is redefined as the demand induced by \(T(\bar{c}_i)\) and \(L(\bar{c}_i)\).

\(^{10}\) Due to the independence assumption, it is not necessary to consider the conditional expectation, \(E_{c_{-i}}[q_i(c_i)]\).
As discussed in the presentation of problems PW and RP, without loss of generality, both definitions can be adjusted when there are opportunity costs. We observe that if the expected utility decreases with \( c_i \), then interim IR is equivalent to imposing a zero profit constraint on the firm with the highest variable cost, i.e., \( V_i(c_{i_{\text{max}}}) \geq 0 \). Ex-post IR is motivated by the observation that interim IR does not guarantee a non-negative pay-off for all opponents’ profiles, \( c_{-i} \). That is, there could be cases where a firm regrets participating. Ex-post IR, therefore, eliminates the risk of ending with a loss. We use the ex-post IR condition to select from the families of solutions to the optimal direct revelation mechanism satisfying interim IR. Hereafter, IR without specification refers to interim IR.

### 3.2.1. Optimal mechanism design: problem formulation

The mechanism design problem, MD, involves \( PW(\cdot) \) maximization, subject to user equilibrium, IC, and IR. Since the government is not able to observe the realization of \( c \) before setting the rules, the objective is written ex-ante. That is, the government optimizes the expected public welfare, based on the prior distribution.

\[
\text{MD} \quad \max_{Q, T, L, M} \mathbb{E}_c \left[ \sum_{i=1}^{I} (q_i(c_i) \cdot CS(T(c_i), N(c_i), L(c_i))) - M_i(c_i) \right] \\
\text{subject to:} \quad \mathbb{P}(N(c_i)) = c(N(c_i), L(c_i)) + T(c_i) \quad i = 1, \ldots, I \\
V_i(c_i) = \mathbb{E}_c [q_i(c_i) \cdot \pi(c_i; T(c_i), N(c_i), L(c_i)) + M_i(c_i)] \quad i = 1, \ldots, I \\
\frac{\partial V_i(c_i)}{\partial c_i} = S_i(c_i) \quad i = 1, \ldots, I \\
S_i(c_i') \leq S_i(c_i) \quad c_i' \leq c_i, \; i = 1, \ldots, I \\
V_i(c_{i_{\text{max}}}) \geq 0, \quad i = 1, \ldots, I \\
c_{i_{\text{min}}} \leq c_i, \; c_i' \leq c_{i_{\text{max}}} \quad i = 1, \ldots, I \tag{16}
\]

where \( CS(\cdot) \) is the consumer surplus function. The objective is to maximize the expected PW(\( \cdot \)), where Eqs. (17)-(21) are, respectively, the user equilibrium, indirect utility, Envelope Theorem, Monotonicity, and IR conditions. Equation set (22) specify the bounds on the variable costs. Also, we note that \( G_i(\hat{c}_i, c_i) = -P_i(\hat{c}_i) \cdot L(\hat{c}_i) \), which means that \( S_i(c_i) = -P_i(c_i) \cdot L(c_i) \).

MD can be re-written by substituting (17) and (18) in the objective function. As is done in Basov (2005) and shown in Appendix A, integration by parts and other steps are used to obtain the following simplified version of the problem, **MD-S**.

\[
\text{MD-S} \quad \max_{Q, T, L} \mathbb{E}_c \left[ \sum_{i=1}^{I} q_i(c_i) \cdot \left( CS(T(c_i), N(c_i), L(c_i)) + \pi(c_i; T(c_i), N(c_i), L(c_i)) - \frac{F(c_i)}{f(c_i)} L(c_i) \right) - V_0 \right] \\
\text{subject to:} \quad \mathbb{P}_i(c_i) \cdot L(c_i) \leq \mathbb{P}_i(c_i') \cdot L(c_i') \quad c_i' \leq c_i, \; i = 1, \ldots, I \\
V_0 = V_i(c_{i_{\text{max}}}) \geq 0, \quad i = 1, \ldots, I \\
c_{i_{\text{min}}} \leq c_i, \; c_i' \leq c_{i_{\text{max}}} \quad i = 1, \ldots, I \tag{23}
\]

We define firm \( i \)'s virtual surplus as

\[
R(c_i; T(c_i), L(c_i)) = SW(c_i; T(c_i), N(c_i), L(c_i)) - \frac{F(c_i)}{f(c_i)} \cdot L(c_i) \tag{27}
\]
Notice that \( R(\cdot) \) captures the \( i \)th firm’s contribution to the expected public welfare given \( T(\cdot), L(\cdot) \), so (23) can be re-written as 

\[
E_{\xi}\left[ \sum_{i=1}^{\mathcal{L}} q_i(c) \cdot R(c; T(c_i), L(c_i)) \right].
\]

Since there is exactly one winner, i.e., \( \sum_{i=1}^{\mathcal{L}} q_i(c) = 1 \), MD-S boils down to the maximization of virtual surplus received from the buyer.\(^{11}\) The optimal allocation rule, therefore, is to award the concession to the bidder that generates greatest virtual surplus. However, \( R(\cdot) \) depends on the prior distribution \( f(\cdot) \), and on the capacity schedule \( L(\cdot) \). The monotonicity constraint further complicates the relationship among \( Q, T \) and \( L \) when maximizing the virtual surplus. Below, we analyze the optimal rules and discuss the relevant assumptions.

### 3.2.2. Optimal mechanism design: problem solution

The definition of the virtual surplus function allows for separation of the optimization of each \( R(\cdot) \) with respect to the toll and capacity schedules, \( T \) and \( L \). From the allocation rule, \( Q \). Expanding the virtual surplus function, we obtain

\[
\begin{align*}
R(c; T, L) &= \int_0^{N(T(c))} P(n)dn - N(T(c), L(c)) \cdot C(N(T(c)), L(c)) - c_0 \\
&\left( c_i + \frac{F(c)}{f(c)} \right) L(c)
\end{align*}
\]

which shows that the objective is to maximize social welfare with a distorted/virtual variable cost, \( \gamma_i(c) \), where \( \gamma_i(c) \equiv c_i + \frac{F(c)}{f(c)} \). The solution to the problem of maximizing the virtual surplus, (28), is obtained from the results to FB with the distorted variable cost. Namely, \( T^P(c) = \tau^*(\gamma_i(c)) \) and \( L^P(c) = K^*(\gamma_i(c)) \). Since \( \gamma_i(c) \geq c_i \), we observe that the distortions from imperfect information lead to (i) higher optimal tolls than in FB, i.e., \( T^P(c_i) \geq \tau^*(c_i) \), (ii) lower optimal capacity, (iii) lower optimal demand, and (iv) more congestion, i.e., higher optimal \( v/c \) ratios.

The optimal allocation rule is to award the concession to the firm with the lowest \( \gamma_i(c_i) \). Unlike the case of complete and perfect information, this need not be the firm with the smallest \( c_i \), unless there is a firm with \( c_i = c_{\min} \), where \( \gamma_i(c) = c_{\min} \) because \( F(c_{\min}) = 0 \). This is the relevant version of the “no distortion at the top” result. Thus, to simplify the analysis, we consider a relevant variation of the Regularity Assumption commonly appearing in the Mechanism Design literature (Che, 1993; Myerson, 1981).

**Assumption 4.** Regularity condition

The virtual unit cost, \( \gamma_i(\cdot) \), is non-decreasing in \( c_i \), i.e., \( \frac{\partial \gamma_i(c)}{\partial c_i} \geq 0 \).

This assumption ensures the ex-post optimality of awarding the franchise to the firm with the lowest virtual unit cost.\(^{12}\) Also, following the discussion in Section 2 and Assumption 4 guarantee that \( T^P(c_i) \) and \( L^P(c_i) \) are increasing and decreasing in \( c_i \), i.e., \( \frac{\partial T^P(c_i)}{\partial c_i} \geq 0 \) and \( \frac{\partial L^P(c_i)}{\partial c_i} \leq 0 \).

Finally, we observe that Assumption 4 ensures that constraint (24) in MD-S is always satisfied. Firms with higher variable costs, \( c_i \), have higher virtual variable costs, \( \gamma_i(c_i) \), and therefore, smaller chance to win the franchise, i.e., \( P_1(c_i) \) is non-increasing with \( c_i \). Also, as discussed above, \( L^P(c_i) \) decreases with \( c_i \). Thus, \( P_1(c_i) \cdot L^P(c_i) \) is non-increasing with \( c_i \).

In terms of the government payments, from (18) and (19) in MD-S, the interim or conditional expected payment is given by

\[
\tilde{M}_i(c_i) = E_{\xi}[M_i(c_i, c_{-i})] = V_0 - \int_{c_{\min}}^{c_i} P_1(x)L^P(x)dx - \int_{c_{\min}}^{c_i} [P_1(c_i) \cdot \pi(c_i; T^P(c_i), N^P(c_i), L^P(c_i))]
\]

Since the IR constraint is binding at optimality, i.e., \( V_0 = 0 \), the first term can be removed from the above expression. The constraints in MD-S, therefore, allow for flexibility in selecting ex-post payment rules as long as their conditional expectations satisfy (29). For example, one may set the ex-post payments to be the same as the interim payments. While attractive because a firm’s payment only depends on its own \( c_i \), not on others, such a rule may violate ex-post IR, i.e., the winner may not receive sufficient payment, while losing firms may receive free payments. Thus, we propose the following payment rule for any realization of \( c \):

\[
M^P_i(c) = -q_i(c) \cdot \pi(c_i; T^P(c_i), N^P(c_i), L^P(c_i)) - \int_{c_{\min}}^{c_i} q_i(x, c_{-i})L^P(x)dx, \quad i = 1, \ldots, L, \ c \in \mathcal{C}_i
\]

\(^{11}\) Even though the virtual surplus is the expected contribution to the public welfare, due to the distortion term, the virtual surplus is different than the actual surplus, i.e., the actual/conditional public welfare given the bidder’s type.

\(^{12}\) We do note that ironing techniques have been developed to address cases where Assumption 4 does not hold (Rochet and Choné, 1998). We also note that the assumption is not as strong as it seems because distributions with non-increasing probability density, such as the uniform and exponential distributions are sufficiently regular. Certain instances of normal and truncated normal distributions satisfy Assumption 4, as well.
where the inequality is due to the fact that \( c_i \leq c_{\text{max}} \) and \( q_i(x, c_{\perp})L^p(x) \) is non-negative. The integral term in the expression is referred to as the information rent. Notice that, for firm \( i \), the integral’s argument is zero, i.e., \( q_i(x, c_{\perp}) = 0 \), until \( x \) becomes the best bid. Because exactly one firm wins, the ex-post information rent is

\[
-\int_{c_{\text{min}}}^{c_{\text{max}}} q_w(x, c_{\perp})L^p(x)dx = \int_{c_w}^{c_{\text{max}}} \pi(x)dx
\]

(31)

where \( w \) and \( l \), respectively, denote the firms with lowest and second lowest variable costs. That is, it is a generalization of the well-known Vickrey Auction (or the second-price auction), where the highest bidder wins, but pays a price equal to the second highest bid, and realizes a value equal to the difference in the bids. Thus, the ex-post government payment is

\[
M^p_w(c) = -\int_{c_{\text{min}}}^{c_{\text{max}}} q_w(x, c_{\perp})L^p(x)dx - q_w(c)\cdot \pi(c_w; T^p(c_w), N^p(c_w), L^p(c_w))
\]

\[
= \int_{c_w}^{c_{\text{max}}} \pi(c_w; T^p(c_w), N^p(c_w), L^p(c_w))
\]

(32)

The first term in the above expression is the profitability difference, and the second term is winner's ex-post operating profit given by:

\[
\pi(c_w; T^p(c_w), N^p(c_w), L^p(c_w)) = N^*(\gamma_w(c_w)) \cdot \tau^*(\gamma_w(c_w)) - C(c_w; K^*(\gamma_w(c_w)))
\]

\[
= N^*(\gamma_w(c_w)) \cdot \tau^*(\gamma_w(c_w)) - C(c_w; K^*(\gamma_w(c_w))) + \frac{F(c_w)}{\int(c_w)} \cdot K^*(\gamma_w(c_w))
\]

\[
= \frac{F(c_w)}{\int(c_w)} \cdot K^*(\gamma_w(c_w)) - c_0 = \frac{F(c_w)}{\int(c_w)} \cdot L^p(c_w) - c_0
\]

(33)

where the last line follows the results presented in Section 2, where we show that the solution to FB leaves the firm with a loss of \( c_0 \). As \( \frac{F(c_w)}{\int(c_w)} \) increases with \( c_w \), the distortion profit increases with the variable cost. Plugging in (33) into (32) yields

\[
M^p_w(c) = c_0 + \int_{c_w}^{c_{\text{max}}} L^p(x)dx - \frac{F(c_w)}{\int(c_w)} \cdot L^p(c_w)
\]

(34)

Thus, in order to make firm participation viable, from an ex-post standpoint, the government must compensate the firm for the fixed costs, \( c_0 \), as well as for the difference between the information rent and the distortion profit. Since the profitability difference depends on the realizations of \( c_w \) and \( c \), it is not possible to establish whether the net payment is greater or lesser than \( c_0 \). When the distortion profit is large enough, the total transfer could be negative, i.e., the firm pays the government.

The optimal solution to MD-S is summarized below:

1. \( Q^p \) is given by \( q^p(c) = \begin{cases} 1 ; & c_i = \min\{c_1, c_2, \ldots, c_l\} \\ 0 ; & \text{otherwise} \end{cases} \)

   A tie-breaking rule may be necessary.

2. \( T^p(c_i) = \tau^*(c_i + \frac{F(c_i)}{\int(c_i)}) \).

3. \( L^p(c_i) = K^*(c_i + \frac{F(c_i)}{\int(c_i)}) \).

4. \( M^p_i(c_i) = -q_i(c) \cdot \pi(c_i; T^p(c_i), N^p(c_i), L^p(c_i)) - \int_{c_{\text{min}}}^{c_{\text{max}}} q_i(x, c_{\perp})L^p(x)dx \).

3.2.3. Optimal mechanism design: balanced budget problem

The analysis presented in the previous subsection can be viewed as generalization of PW to account for imperfect information. In this subsection, we consider a structured counterpart to RP; that is, an extension where the government makes no payments. Due to the Revelation Principle, the only adjustment needed is to impose an additional budget constraint on the set of incentive compatible mechanisms. The new problem is

\[
\text{BBMD} \quad \max_{Q, T, L} \quad E_c \left[ \sum_{i=1}^{l} q_i(c) \cdot \left( SW(c_i, T(c_i), L(c_i)) - \frac{F(c_i)}{\int(c_i)} L(c_i) \right) - V_0 \right]
\]

subject to:

\[
P_i(c_i) \cdot L(c_i) \leq P_i(c_i') \cdot L(c_i') \quad i = 1, \ldots, l
\]

(35)

(36)
\[ E_{c_{i}}[M_i(c_i, c_{-i})] = V_0 - \int_{c_{\text{max}}}^{c_i} \mathbb{P}_1(x) L(x) dx - \mathbb{P}_1(c_i) \pi(c_i; T(c_i), N(c_i), L(c_i)), \quad i = 1, \ldots, I \]  

(37)

\[ \sum_{i=1}^{I} M_i(c) \leq 0 \]  

(38)

\[ V_0 = V(c^{\text{max}}) \geq 0, \quad i = 1, \ldots, I \]  

(39)

\[ c_{\text{min}} \leq c_i, \quad c_i' \leq c_{\text{max}}, \quad i = 1, \ldots, I \]  

(40)

Constraint (37) combines (18) and (19) from MD, and provides an explicit definition of the government payments. Constraint (38) restricts the ex-post payments to being non-positive. The proposition below presents a solution to BBMD that relies on decomposing the problem into 2 parts: the allocation rule, and the toll-capacity schedule.

**Proposition 3. Optimal balanced budget mechanism**

Under a regular, common prior, the following direct mechanism optimizes BBMD:

1. \( Q^B \) is given by \( q_i(c) = \begin{cases} 1 & ; \quad c_i = \min \{ c_1, c_2, \ldots, c_I \} \\ 0 & ; \quad \text{otherwise} \end{cases} \)

2. Payment is given by,

\[ M_i^B(c) = \begin{cases} -q_i(c) \cdot \left[ \pi(c_i; T^B(c_i), N^B(c_i), L^B(c_i)) + \frac{\int_{c_i}^{c_{\text{max}}} \mathbb{P}_1(x) L(x) dx}{\mathbb{P}_1(c_i)} \right] & ; \quad \mathbb{P}_1(c_i) \neq 0 \\ 0 & ; \quad \text{otherwise} \end{cases} \]

3. For each firm \( i, \) \( T^B(c_i) \) and \( L^B(c_i) \) solve the following Ramsey Problem RP-R:

\[ \text{RP - R} \quad \max_{\gamma, K} \quad \int_0^{N} P(n) dn - N \cdot C(N, K) - C(\gamma(c_i); K) \]

subject to:

\[ P(N) = \tau + C(N, K) \]

\[ \mathbb{P}_1(c_i) \cdot (N \cdot \tau - C(c_i; K)) \geq -\int_{c_{\text{min}}}^{c_i} \mathbb{P}_1(x) L^B(x) dx \]

The derivations appear in Appendix A. Recall that for MD-S we propose an ex-post payment rule, \( M_i^P, \) which depends on the lowest and second-lowest variable costs. In contrast, \( M_i^B, \) is not contingent on other bidders’ types. The reason is that ex-post budget balance requires the total payment to be non-positive for any realization of \( c. \) This is equivalent to restricting the payment that the winner receives. Given the General Pay-off Equivalence, the interim payment, i.e., the conditional expectation given the winner’s type, is fixed. Thus, to restrict the winner’s ex-post payment, we use the constraint in RP-R to set all of the ex-post payments to their expectation. The difficulty arises, however, from the computing of the conditional expectation, since it not only depends on the prior but also on the capacity rule \( L^B(\cdot). \) In practice, BBMD can be solved recursively, i.e., alternating between optimal toll/capacity schedules and the constraint in RP-R. To conclude, we note that BBMD is not just restricted to zero-budget case. We can apply it to any fixed budget (by adjusting \( c_0). \)

**4. Implementation**

In the preceding section, we present direct revelation mechanisms that optimize public welfare. However, in many situations it may be difficult or unappealing for firms to evaluate or disclose their variable costs, e.g., they may want to keep their business practices private. Thus, building on Guesnerie (1981); Rochet (1985); Che (1993), we consider 3 implementations that circumvent the need to disclose the private variable costs. In these implementations, the government specifies a scoring function and a payment rule in advance of the auction. Firms bid based on their variable costs.

In the first implementation, we construct a scoring function, for a first-score auction, that leads firms to place bids that maximize the virtual surplus generated by the project, and thus, leads to an implementation of the optimal direct revelation mechanism. We then analyze a naïve scoring auction, where the scoring function corresponds to the public welfare, i.e., the government’s objective. The implementation is sub-optimal because, unlike the virtual surplus, the public welfare ignores distortions. Finally, we present a demand pricing mechanism where firms select bundles consisting of a demand level and an associated payment, with the firm that bids the largest demand winning the auction. Because the mechanism partially accounts for distortions, we show analytically that it performs at least as well as the naïve scoring auction.
4.1. Optimal scoring auction

Scoring auctions are analogous to first or second price auctions. Rather than price, the criterion to evaluate bids consists of a function that balances tradeoffs along multiple dimensions. Scoring auctions are widely used in competitive bidding for public projects, with the aforementioned A+B bidding being a representative example. Scoring auctions are categorized into first-score, second-score, and second-preferred auction. The first 2 are analogous to the first-price and second-price auction, where the winning firm, the bidder with the highest score, is required to set the technical parameters, e.g., toll, capacity, payment, to match the best or the second-best score, respectively. The second-preferred auction requires the winning firm to set the parameters to those appearing in the second-best bid. Che (1993) shows that optimal direct revelation mechanisms can be implemented as first or second score auctions with different compensation strategies. This, however, is not possible with second-preferred auctions. In this section, we derive a scoring function that leads to the implementation of the optimal direct revelation mechanism as a first-score auction. The scoring function is structured as described below in Assumption 5.

Assumption 5. Structure of scoring function

The scoring function is assumed to have the form $\alpha(\tau, K, M) = \alpha(\tau, K) - M$, where the arguments correspond to the toll, capacity, and government payment. $\alpha(\cdot)$ is referred to as the quality scoring function.

Each bidder trades off the potential profit with the probability of winning based on their score. For the first-score auction, each bidder solves

$$
\max_{\tau, K, M} \{ \pi(c_i; \tau, N(\tau, K), K) + M \} \cdot P(\text{winning with } \alpha(\tau, K, M)) \Leftrightarrow
\max_{\tau, K, M} \{ \pi(c_i; \tau, N(\tau, K), K) + M \} \cdot [F_0(\alpha(\tau, K, M))]^{1-1}
$$

where $F_0(\cdot)$ is the cumulative distribution of scores at equilibrium. To understand $F_0(\cdot)$, we consider the Bayesian Nash Equilibrium, at which each firm applies a symmetric strategy that maps its variable cost to a score. Because production efficiencies are iid, and the equilibrium strategy is symmetric, the distribution of scores is also iid. The probability that a firm wins is the joint probability that its $I - 1$ opponents have lower scores, i.e., higher unit costs, and is given by $F_0(\cdot)^{I-1}$. Desirable specifications of the scoring function lead to (i) (equilibrium) scores that increase with production efficiency, i.e., $\alpha(\tau, K, M)$ decrease with $c_i$; and (ii) induce toll and capacity levels that maximize the expected public welfare. Lemma 1 is an intermediate result that we use in the specification of scoring functions that satisfy the second condition.

Lemma 1. Optimal toll and capacity levels in scoring auctions

With a scoring function $\alpha(\tau, K, M) = \alpha(\tau, K) - M$, firm $i$ chooses toll and capacity levels, $\tau$ and $K$, that maximize $\alpha(\tau, K) + \pi(c_i; \tau, N(\tau, K), K)$.

The proof is presented in Appendix A. Lemma 1 provides a framework to design (quality) scoring functions, $\alpha(\cdot)$, that induce firms to bid (based on) toll and capacity levels that maximize virtual surplus, $T^\pi(c_i)$ and $L^\pi(c_i)$. Proposition 4 follows Che (1993) and relies on the observation that the toll and capacity levels that optimize the bidding strategy also maximize the virtual surplus function for a given variable cost, $c_i$.

Proposition 4. Optimal scoring function

A first-score auction with the quality scoring function below results in an implementation of the optimal direct revelation mechanism.

$$
\alpha^p(\tau, K) = CS(\tau, N(\tau, K), K) - \int_0^K \frac{F(c^p(k))}{f(c^p(k))} dk
$$

where $c^p(\cdot)$ is the inverse of the public optimal capacity schedule, $L^p(c_i)$. Furthermore, adding the constraint $M \leq 0$ and adjusting the inverse schedule results in an implementation of the optimal balanced budget mechanism.

See Appendix A for the proof. Given the specification of $\alpha^p(\tau, K)$. $\alpha^p(\tau, K) + \pi(c_i; \tau, N(\tau, K), K) = SW(c_i; K, N(\tau, K)) - \int_0^K \frac{F(c^p(k))}{f(c^p(k))} dk$, which is similar to the virtual surplus function, $R(\cdot) - both contain the social welfare and a term of the inverse hazard rate. Further, we note that both functions have the same derivatives with respect to $\tau$ and $K$. Thus, given a firm’s variable cost, $c_i$, toll and capacity levels that maximize $R(\cdot)$ also maximize $\alpha^p(\cdot) + \pi(c_i; \cdot)$.  

4.2. Naive scoring auction

Intuitively, considering a naive scoring function, where $\alpha^p(\tau, K) = CS(\tau, N(\tau, K), K)$, seems appealing. That is, a rule where each firm’s score is a function of the ex-post consumer surplus generated by its plan. The naive rule, where awards are exclusively based on the government’s preference, is widely-used in procurement auctions. In contrast to $\alpha^p(\cdot)$, $\alpha^p(\cdot)$ ignores the distortion term. Again, following Lemma 1, each firm maximizes $\alpha^p(\tau, K) + \pi(c_i; \tau, N(\tau, K), K) = $
CS (τ, N(τ, K), K) + π (c; τ, N(τ, K), K) = SW (c; τ, N(τ, K), K) and offers F∗(c) and K∗(c). From the analysis under complete and perfect information in Section 2, it follows that non-distorted toll-capacity schedules yield higher ex-post social welfare than do the distorted schedules. With imperfect information, however, the non-distorted schedules reduce the expected public welfare. Following Che (1993):

**Lemma 2. Ex-ante public welfare of scoring auction**

In the first-score auction, the expected public welfare is given by

\[ E_w\left[ SW(c_w; \tau_w, N(\tau_w, K_w), K_w) - F(c_w)K_w\right] \]

where \( \tau_w, K_w \) are the toll and capacity choice of the winner \( w \).

The proof is similar to the proof of Proposition 1. While under the optimal scoring rule each firm optimizes the virtual surplus it generates, under the naïve rule each firm optimizes the \( SW(\cdot) \) it generates. Letting \( R^*(x) \) denote the optimal social welfare generated by a firm with variable cost \( x \), i.e.,

\[ R^*(x) = \max_{\tau, K} \int_0^N P(n)dn - C(N, K) - C^*(x; K) \]

Thus, for firm \( i \), the respective virtual surplus from the naïve and optimal scoring rules are \( R^*(c_i) - \frac{F(c_i)}{f(c_i)}K^*(c_i) \) and \( R^*(\gamma_i) \). The difference is given by,

\[ \Delta_n = R^*(\gamma_i(c_i)) - R^*(c_i) + \frac{F(c_i)}{f(c_i)}K^*(c_i) = \int_{c_i}^{\gamma_i(c_i)} R^*(h)dh + \frac{F(c_i)}{f(c_i)}K^*(c_i) \]

where the last line follows from the Envelope Theorem. Notice the loss is always non-negative because \( K^*(\cdot) \) is a strictly decreasing function. There is no loss of virtual surplus when firm \( i \) has \( c_i = c_{\min} \) because there is no distortion. The loss comes from the variation of optimal capacity choices for variable costs of either \( c_i \) or \( \gamma_i(c_i) \). Thus, the loss is small when the capacity schedule is not sensitive to changes in variable cost and the factor, \( \frac{F(c_i)}{f(c_i)} \), is small. The loss is realized through the government payment, \( M \). In the naïve scoring auction, a firm’s ex-post operating profit \( \pi(c_i; \tau^*(c_i), N^*(c_i), K^*(c_i)) = -c_0 \). Recalling that \( PW(\cdot) = SW(\cdot) - \pi(\cdot) - M = CS(\cdot) - M \), from Lemma 2, the ex-post payment, \( M \), satisfies \( c_0 - M = -\frac{F(c_i)}{f(c_i)}K^*(c_i) \).

and the ex-ante payment equal to \( E\left[ \frac{F(c_w)}{f(c_w)}K^*(c_w)\right] + c_0 \). Comparing the ex-ante firm profit and government payment to expressions (33) and (34) for the optimal mechanism, we observe that there is no distortion profit. Even though the choices in the naïve scoring auction generate more social welfare, the (much) larger government payment results in a smaller public welfare than those obtained with the optimal mechanism. This observation highlights the role of the government payment in the optimal mechanism as a tool to limit the firm’s ability to exploit information asymmetries.

This observation highlights the role of distortions in the mechanism to limit the information rent at the expense of a loss in consumer surplus from the implementation of sub-optimal ex-post toll and capacity levels.\(^{14}\) Indeed, following Tan et al. (2010), we observe that the \( v/c \) ratio stemming from \( f^P(c_i), L^P(c_i) \), \( \mu^*(\gamma_i(c_i)) \) is ex-post Pareto inefficient, which means that, after the award, both the concessionaire and the government have incentive to renegotiate the agreement and set the ratio to \( \mu^*(c_i) \). The government, however, must specify the mechanism ex-ante. Furthermore, renegotiation opportunities, may lead to inconsistencies with the IC restriction, i.e., firms may have incentive to lie.

4.3. A demand pricing mechanism

Inspired by the maximum patronage auction of Verhoef (2007), we present a demand pricing mechanism, which is motivated by the complexity of the scoring functions presented above. The proposed mechanism is based on the observation that the public welfare can be written as follows:

\[ PW(\cdot) = CS(\cdot) - M = \int_0^N P(n)dn - N \cdot C(N, K) - N\tau - M \]

\[ = \int_0^N P(n)dn - N \cdot C(N, K) + \tau - M = \int_0^N P(n)dn - N \cdot P(N) - M \]

where the last expression follows from the user equilibrium condition. In turn, this shows that, no matter how a firm sets the toll and capacity, the public welfare is determined by the government payment, \( M \), and the demand, \( N \). Thus, the

\(^{14}\) In this paper, the Regularity Condition in Assumption 4 ensures that the information structure does not distort concessionaire selection, which, in general, may reduce consumer surplus further. It is also pertinent to mention that Assumption 4 ensures IC.
government’s task is to assess the tradeoffs between demand and the payment, whereas the firm is tasked with operating efficiency. In particular, the user equilibrium condition reduces the firm’s task to optimizing the capacity level to satisfy a given demand. This is seen by rewriting the firm’s operating profit as
\[
\pi(\cdot) = N \tau - C^v(K) = N \cdot (P(N) - C(N, K)) - C_i'(K)
\]
For a fixed demand level, the first-order condition is given by
\[
\frac{\partial \pi(\cdot)}{\partial K} = -N \cdot C_K - c_i = 0 \Rightarrow 
\mu^* : g' (\mu) \mu^2 \bigg|_{\mu = \mu^*} = c_i
\]
which coincides with the result for FB in Eq. (5). Thus, if bids consist of demand and payment combinations, firms are competing by adjusting the capacity level to the optimal \(v/c\) ratio given in Eq. (42). For a given \(N\), firm \(i\)'s profit is
\[
\pi^d(c_i; N) = N \cdot (P(N) - g(\mu^*(c_i))) - c_0 - c_i \cdot \frac{N}{\mu^*(c_i)}
\]
where \(\mu^*(c_i)\) is the \(v/c\) ratio for the (undistorted) variable cost \(c_i\).

Inspired by the irrelevance of toll and capacity levels on the public welfare, we present a simple mechanism, in which a government publishes a compensation policy contingent on demand levels submitted by bidders, and denoted \(Y : \mathbb{R}^+ \mapsto \mathbb{R}\). Firm selection is based on the demand proposals. Understanding that the government’s response is given by \(Y\), we define firm \(i\)'s bidding strategy as the mapping \(N^d : \mathcal{C} \mapsto \mathbb{R}^+\). Then, for a firm with variable cost \(c_i\), its equilibrium strategy is to bid the demand at \(N^d(c_i)\), and receive payment \(Y(N^d(c_i))\) if it wins. Following the Revelation Principle, we consider the equivalent direct mechanism, where each firm reports its type \(c_i\), thereby committing to a demand and payment, respectively, following \(N^d\) and \(M^d : \mathcal{C} \mapsto \mathbb{R}\). Here, \(M^d\) is a function of the bidder’s own type, \(c_i\), as opposed to the set, \(\mathcal{C}\). That is, the payment is assumed to be independent from other bidders’ types as the demand pricing schedule, \(Y\), is a function of the bidder’s demand proposal only. Repeating our construction yields the following optimal demand pricing problem:
\[
\text{DP} \quad \max_{Q, K, M} \quad E_c \left[ \sum_{i=1}^{I} q_i(c_i) \cdot (CS(N(c_i)) - M(c_i)) \right]
\]
subject to:
\[
V_i(c_i) = \mathbb{P}(c_i) \cdot \left( \pi^d(c_i; N(c_i)) + M(c_i) \right), \quad i = 1, \ldots, I
\]
\[
\bar{R}(c_i) = \frac{N(c_i)}{\mu^*(c_i)}, \quad i = 1, \ldots, I
\]
\[
\frac{\partial V_i(c_i)}{\partial c_i} = S_i(c_i) = -\mathbb{P}(c_i) \cdot \bar{R}(c_i), \quad i = 1, \ldots, I
\]
\[
S_i(c_i') \leq S_i(c_i), \quad c_i' \leq c_i, \quad i = 1, \ldots, I
\]
\[
V_i(c_{\text{max}}) \geq 0, \quad i = 1, \ldots, I
\]
\[
c_{\text{min}} \leq c_i, \quad c_i' \leq c_{\text{max}}, \quad i = 1, \ldots, I
\]
\[
\text{DP} \quad \text{is different than MD in that: (i) firms select the capacity schedule, } \bar{R}, \text{ through the non-distorted } v/c \text{ ratio, } \mu^*(c_i); \ (ii) The toll schedule is removed because it is determined by the capacity and demand schedules; (iii) the payment in the indirect utility expression, Eq. (45), appears inside the parenthesis, which means that payments are restricted to the winning firm; and (iv) critically, we note that the monotonicity condition, Eq. (48), depends on each firm’s optimal \(v/c\) ratio, \(\mu^*(c_i)\).

Assumption 4 ensures the monotonicity of the capacity schedule, and therefore that the monotonicity constraint in MD-S is satisfied. In DP, however, the assumption is not sufficient to ensure the monotonicity of \(\bar{R}\), and thereby the monotonicity of \(S_i(c_i)\). Below, we first derive the optimal demand schedule, i.e., optimal bidding strategies. We then present a condition that ensures the monotonicity of the solution.

As in our solution to MD-S, the payment is given by (45), which we substitute into the objective function. This allows for separate consideration of firm selection and virtual surplus maximization. Firm \(i\)'s virtual surplus is given by
\[
R^d(c_i) = \int_0^N P(n)dn - N \cdot (\mu^*(c_i)) - C^v(\gamma_i(c_i); \bar{R}(c_i))
\]
The optimal demand schedule, \( N^d(c_i) \), is obtained by evaluating the first order condition
\[
N^d(c_i) : P(N)|_{N=N^d(c_i)} = \frac{\gamma_i(c_i)}{\mu^*(c_i)} + g(\mu^*(c_i))
\]
where we observe that the distortion on choices of \( N \) does not affect \( \mu^*(c_i) \). The expression for the distorted profit at the optimal choice of demand, \( N^d(c_i) \) is
\[
\pi^d(c_i; N^d(c_i)) = N^d(c_i) \cdot (P(N^d(c_i)) - g(\mu^*(c_i))) - C(c_i; \frac{N^d(c_i)}{\mu^*(c_i)})
\]
\[
= \frac{N^d(c_i)}{\mu^*(c_i)} (\gamma_i(c_i) - c_i) - c_0
\]
\[
= k^d(c_i) \cdot \frac{F(c_i)}{f(c_i)} - c_0
\]
The first term in the final expression is the product of the adjusted capacity and the distorted variable cost. This is similar to the result for MD-S. From (51), we observe that a government always prefers the firm with the lowest \( \gamma_i(c_i) \) because it yields the highest virtual surplus, even when multiple firms serve the same demand level. Unfortunately, even with the Regularity Assumption, optimal demand schedules, satisfying Condition (51), are not always monotonic in \( c_i \). To see this, we write \( N^d(c_i) \) using the demand function, \( P^{-1}(-) \), i.e., \( N^d(c_i) = P^{-1} \left( g(\mu^*(c_i)) + \frac{\gamma_i(c_i)}{\mu^*(c_i)} \right) \), and take the derivative with respect to \( c_i \):
\[
\frac{\partial N^d(\cdot)}{\partial c_i} = \frac{1}{P'} \left( g' \mu' + \frac{\gamma'_i \mu' - \gamma \mu'}{(\mu^*)^2} \right) = \frac{1}{P' \cdot (\mu^*)^2} \left( g' \mu' (\mu^*)^2 + \gamma'_i \mu^* - \gamma \mu' \right)
\]
where the second line follows from (42). The sign of the derivative, which depends on the prior distribution and on the congestion function, is inconclusive. Thus, to arrive at a regularity condition that ensures monotonicity of the demand, we assume that the condition presented in Lemma 3 below holds for all firms.

**Lemma 3. Regularity Condition for Demand Pricing Mechanism**

For any congestion function, \( g(\cdot) \), \( N^d(c_i) \) is decreasing with respect to \( c_i \) if and only if
\[
2c_i \gamma'_i(c_i) \geq \frac{F(c_i)}{f(c_i)}, \quad c_i \in C, i = 1, \ldots, I
\]

The proof is in Appendix A. We note that the condition specified in Lemma 3 is not necessary for some congestion functions. For example, for a uniform distribution from \( c_i^{\min} \) to \( c_i^{\max} \), where \( c_i^{\min} \geq 0 \), the left-hand-side in the condition equals 4 \( c_i \), while the right-hand-side equals \( c_i^{\min} \). Thus, any uniform distribution with non-negative support ensures the monotonicity of \( N^d(c_i) \). We also note that the condition in Lemma 3 is stronger than Assumption 4. The latter requires \( \gamma'_i \) to be non-negative, whereas the former restricts \( \gamma'_i \) to be larger than a positive number. We note that only prior distributions with high kurtosis violate the stronger condition. The following proposition shows that the result ensures that the monotonicity constraint in DP holds.

**Proposition 5. Monotonicity Condition**

\( N^d(c_i) \) decreasing with respect to \( c_i \) ensures that the monotonicity condition in DP, Eq. (48), is satisfied.

See Appendix A for the proof. The optimal payment rule is obtained by substituting the Envelope Theorem conditions, Equation set (47), into Eqs. (45) as shown below:
\[
M^d(c_i) = \frac{V_i(c_i)}{P_i(c_i)} - \pi^d(c_i; N^d(c_i)) = \frac{\int_{c_i^{\min}}^{c_i^{\max}} P(x) \bar{R}(x) dx}{P(c_i)} - \pi^d(c_i; N^d(c_i))
\]
\[
= \frac{\int_{c_i^{\min}}^{c_i^{\max}} P(x) \bar{R}(x) dx}{P(c_i)} - \frac{N^d(c_i)}{\mu^*(c_i)} \frac{F(c_i)}{f(c_i)} c_0
\]
where the last expression follows from Eqs. (51) and (43).

In the context of DP, the Taxation Principle ensures that the Revealed Mechanism can be implemented through a non-linear pricing model, as long as all bids at a given demand level commit to the same payment (without regard to the underlying \( c_i \)) (Rochet, 1985). If \( N^d(c_i) \) is strictly decreasing in \( c_i \), then each demand level corresponds to a unique variable cost, which satisfies the condition. If, however, \( N^d(c_i) \) is not strictly decreasing, there could be different payments for the same demand level. In particular, firms with lower variable costs would request higher payments to match less efficient firms, which would preclude implementation of the optimal demand pricing mechanism. This, in turn, explains the importance of Lemma 3 to enable governments to screen firms based on their demand proposals. The payment, \( M^d \), as a function
of the demand, $N^d$ is written as

$$Y(N) = M^d(c^d(N^d(c_i))) = \frac{\int_{\min}^{\max} P(x)K^d(x)dx}{\int P(c^d(N^d(c_i)))} - K^d(c_i) \cdot \frac{N(c_i)}{\int P(c^d(N^d(c_i)))} + c_0$$

(53)

where $c^d(\cdot)$ is the inverse of the monotonically decreasing demand function, $N^d$. The first, second, and third terms, respectively, represent the information rent, the excess distortion profit, and the fixed cost. Again, the government recovers the distortion profit, which reduces the payment. Fig. 1 presents an example of pricing schedules. The parameters used in the example are presented in Section 5.

The information rent is always positive, and increasing with demand. Since both the distortion and demand decrease with variable costs, the distortion profit decreases with the demand. As a result, the government payment, increases with demand. It is not a coincidence that the payment curve in the figure transitions from negative to positive. The reason is that, for low demands, i.e., high variable costs, the distortion profit, which the government recovers in the payment, is high, but the information rent is low. In contrast, for high demands the distortion profit is small, but the information rent is high. At the high end, the government compensates the firm at an amount equal to its information rent.

In summary, the optimal mechanism is given as

**Proposition 6. Optimal demand pricing mechanism**

The government implements the optimal demand pricing mechanism, DP, by offering $Y$, as specified from Eq. (53), and selecting the proposal with the highest demand.

The proof appears in Appendix A. Implementation of the demand pricing mechanism is simple and transparent because bidders compete along one dimension, i.e., demand, thereby avoiding evaluation of complicated scoring functions, for example. There is, however, a caveat in the demand pricing mechanism stemming from the fact that demand is distorted, but the v/c ratios are not, and as a result, the mechanism does not optimize the virtual surplus. To see this, we define $R^*(x, y)$ as the optimal aggregate surplus associated with an optimal v/c ratio for a variable cost of $x$, and for a virtual variable cost $y$, i.e.,

$$R^*(x, y) = \max_N \int_0^N P(n)dn - N \cdot g(\mu^*(x)) - C\left(\gamma, \frac{N}{\mu^*(x)}\right)$$

In MD-S, firm $i$ maximizes the virtual surplus that it generates, $R^*(\gamma_i, y_i)$, whereas in DP, each firm maximizes $R^*(c_i, y_i)$. From the Envelope Theorem

$$\frac{\partial R^*(x, y)}{\partial x} = \left(\frac{\gamma_i}{\mu^{*2}(x)} - g'(\mu^{*}(x))\right) \cdot N^c(x, y) \cdot \mu^{*2}(x) \cdot \frac{N^c(x, y) \cdot \mu^{*2}(x)}{(\mu^{*}(x))^2}$$

where $N^c(x, y)$ is the argument that maximizes $R^*(x, y)$. From (5), we see that $\gamma_i$ is the unique point where $\frac{\partial R^*(x, y)}{\partial x} = 0$. Because both $\mu^*(x)$ and $g'(\mu^*(x))$ are increasing, $\frac{\partial R^*(x, y)}{\partial x} > 0$ for $x < \gamma_i$, and $\frac{\partial R^*(x, y)}{\partial x} < 0$ for $x > \gamma_i$. Thus,

$$\Delta_d = R^*(\gamma_i, y_i) - R^*(c_i, y_i) = \int_{\gamma_i}^{y_i} \left(\gamma_i - g'(\mu^*(h)) \cdot \mu^{*2}(h)\right) \cdot \frac{N^c(h, y_i) \cdot \mu^{*2}(h)}{(\mu^*(h))^2} dh > 0$$

(54)
which shows that there is always a loss from not distorting the v/c ratio. However, in most cases, the loss is small. A bound for the loss is derived from (54) and (5)

\[
\Delta_d = \int_{c_i}^{N} (\gamma_i - h) \cdot \frac{N^v(h, \gamma_i) \cdot \mu^*(h)}{\mu^*(h)^2} dh < (\gamma_i - c_i) \cdot \int_{c_i}^{N} \frac{N^v(h, \gamma_i) \cdot \mu^*(h)}{\mu^*(h)^2} dh \\
= \frac{F(c_i)}{f(c_i)} \int_{\mu^*(c_i)}^{\mu^*(N)} \frac{N^v(m, \gamma_i)}{m^2} dm < \frac{F(c_i)}{f(c_i)} \cdot N^v(c_i, \gamma_i) \left( \frac{1}{\mu^*(c_i)} - \frac{1}{\mu^*(\gamma_i)} \right)
\]

That is, the closer the inverse of two v/c ratios are, the tighter bound on the virtual surplus loss. Hence, \(\Delta_d\) is expected to be small when the distortion \(\frac{f(c_i)}{f(c_i)}\) is small or the optimal v/c ratios are large.

To conclude this section, we observe that the virtual surplus loss from implementation of the demand pricing mechanism is always less than the loss stemming from implementation of the commonly-used naïve scoring rule, i.e., \(\Delta_n - \Delta_d \geq 0\). The result appears as Lemma 4 in Appendix A. The intuition is that the partial distortion, i.e., the demand distortion in the demand pricing mechanism, is better than the “no distortion” situation with the naïve scoring auction. Distorting demand is not sufficient to maximize ex-ante public welfare, but the distortion is second-best under the constraint that bidders self-select the capacity levels. Because firms self-select, and following the discussion from the previous section, the resulting v/c ratio is ex-post optimal. The numerical examples, in Section 5, show that \(\Delta_d\) is close to 0, and significantly lower than the loss from the naïve scoring rule, \(\Delta_n\).

5. Numerical examples and discussion

We present numerical examples to compare the mechanisms presented in the previous sections, and provide insights into their relative performance. The variable costs are drawn from a truncated normal density function, \(f(\cdot)\), with mean and standard deviation of $30,000 and $3,000 per veh/hour, respectively. The support of \(f(\cdot)\) is between $20,000 per veh/hour to $40,000 per veh/hr, the fixed cost is set to $2 million. The congestion function is assumed to take the form of BPR function, specifically, that \(C(N, K) = 1 \cdot 66 \cdot \left[1 + 0.15 \cdot \left(\frac{N}{K}\right)^4\right]\).\(^{15}\) The inverse demand is assumed to be linear as \(P(N) = 136 - 0.05N\) (in $).

The toll and capacity schedules in the optimal direct revelation mechanism and scoring auction, presented in Sections 3.2.2 and 4.1, maximize virtual surplus for each variable cost, \(c_i\). Fig. 2 shows the virtual surplus generated by the mechanisms as a function of \(c_i\). The mean variable costs were used to specify the parameters for the auction with predetermined toll and capacity levels, i.e., \(\hat{c} = $30,000 per veh/hour. We observe that, for all mechanisms, the virtual surplus is decreasing reflecting the loss of social welfare associated with awarding concessions to less efficient firms. \(\Delta_n\), the difference between the virtual surplus generated by the optimal and naïve mechanisms increases with \(c_i\) because the distortion profit increases. There is also a small correction corresponding to the difference in social welfare generated by the corresponding capacity levels. Following Lemma 4, we observe that in demand pricing mechanism, much of the distortion profit is recovered through the demand distortion, which is why \(\Delta_d\) is small. In the predetermined mechanism there is an

\[^{15}\) The coefficient reflects a highway with 66 min free-flow travel time, and an estimated user value of time of $1 per minute.

!!Fig. 2. Virtual surplus comparison.!!
The virtual surplus is an auxiliary function that corresponds to the conditional contribution to the ex-ante public welfare, given \( c_w \). That is, the ex-ante public welfare is a weighted sum of the virtual surplus functions displayed in Fig. 2. The weights derive from the probability density function for the set of variable costs, \( \phi(\cdot) \), which, in turn, depends on the number of firms participating in the auction, \( I \). To illustrate this dependence, Fig. 3 presents a comparison of ex-ante public welfare obtained with the different mechanisms as a function \( I \).
We observe that the *ex-ante* public welfare increases with competition. This is intuitive and follows from the facts that the variable costs for the best and second best firms, \( c_w \) and \( c_i \), are such that \( c_w - c_{\text{min}} \) and \( c_i - c_w \) are more likely to be smaller with increased competition. The former means that the expected consumer surplus, \( CS(\cdot) \), is larger;\(^{16}\) and the latter means that information rent, and thus, the expected payment that the winning firm receives is smaller. The “no distortion at the top” result together with the facts explain why the *ex-ante* loss in public welfare associated with the naïve scoring rule converges to 0 as \( I \) increases. This is shown in Fig. 3(b). Following earlier discussion in the context of Fig. 2, we note that optimal toll and capacity schedules are fixed and set independently of \( I \). Thus, the effect of competition stems from self-adjustment of the mechanisms.

One of the limitations of considering expectations as performance criteria is that actual or *ex-post* performance may be poor for certain realizations of the random variable(s). Thus, here we assess the objective involving maximization of the *ex-ante* public welfare as a function of the winning firm’s variable cost, \( c_w \). In particular, Fig. 4 presents the conditional expected public welfare given \( c_w \). We refer to this function as the actual surplus to distinguish it from the virtual surplus. The government payments also depend on \( c_i \), i.e., from (34), the optimal *ex-post* government payment is given by \( M^G_{cw}(c_i) = c_0 + \int_{c_w}^{c_i} L^p(x)dx - \frac{P(c_w)}{P(c_i)}. \) \( L^p(c_w) \). Thus, to obtain the actual surplus, we subtract the conditional expected payment from the deterministic consumer surplus, given \( c_w \). Again, \( c_i \) depends on \( \phi(\cdot) \), which, in turn, depends on \( I \).

For the single bidder case, the naïve and predetermined mechanisms yield constant expected public welfare levels reflecting the bargaining positions of the government and the firm. The levels depend on the government’s outside option, which we assume involves building the road that maximizes \( PW \) for a variable cost set at the upper bound of \( c_{\text{max}} \), i.e., we set \( c_i = c_{\text{max}} \). The payment leaves the government with a surplus equal to its outside option. Hence, more efficient firms receive larger payments, and the less efficient firms receive smaller ones. Since the predetermined mechanism does not self-adjust, it performs worse than the naïve scoring rule. The performance of the optimal and demand pricing mechanisms is similar. Both mechanisms yield higher expected public welfare levels when \( c_w \) is low, but lower levels when \( c_w \) is high, than those of the other 2 mechanisms. This shows that no mechanism is optimal across all realizations of \( c_w \). In the example, we observe that the optimal and demand pricing mechanisms perform poorly in low-likelihood scenarios. This observation is even more apparent as \( I \) increases, which increases the likelihood of lower \( c_w \). In turn, the threshold, at which the optimal and demand pricing mechanisms are outperformed decreases.

### 6. Summary and conclusions

We use the framework of Myerson (1981) to analyze road concession auctions with imperfect information. In particular, we characterize direct revelation mechanisms that maximize expected public welfare for cases with and without restrictions on government payments. Our analysis shows that the optimization problem can be separated into 2 subproblems: firm selection and virtual surplus maximization. The solution to the former is to award concessions to the firm with the lowest (virtual) variable cost. Optimal toll and capacity schedules for the virtual surplus maximization problem correspond to instances of social welfare maximization problems with perfect information, but with distorted variable costs. Such schedules capture firms’ incentives to exploit private information. Namely, inefficient firms attempt to profit at the expense of consumer/user surplus by distorting their variable costs, thereby proposing roads with lower capacities and higher tolls. On the other hand, efficient firms have little incentive to distort, and instead attempt to exploit their market position and competitive advantages. The assumption of complete information allows for auctions where governments can “recover” profits stemming from distortions. In contrast, profits stemming from relative market position, i.e., the information rent, cannot be recovered.

Following the framework of Che (1993) for the analysis and design of procurement auctions with quality variables, we derive a scoring function for a first-score auction that leads to the implementation of the optimal direct revelation mechanism. The function includes a term that controls for firms’ incentives to distort. We also consider the performance of 2 simple auctions: a naïve scoring auction where bids are scored based on the social welfare that they generate, and a demand pricing mechanism where, as in Verhoef (2007), the bid selection criterion is demand/patronage maximization. We show that the simple auctions are sub-optimal because they lead to virtual surplus losses, for which we present expressions. In the case of the naïve scoring auction, the tolling and capacity schedules are those obtained with perfect information, and thus, generate greater total surplus than the corresponding schedules in the optimal mechanism. The government payments, however, are much larger, which accounts for the virtual surplus reduction. The intuition behind the patronage maximization criterion is that, for a given demand function, consumer surplus is determined by the demand served. For a given demand level, therefore, firms compete on the capacity they set, and thus the congestion they induce, which is determined by the profit and social welfare maximizing \( v/c \) ratio, \( \mu^*(c_i) \). As in the optimal mechanism, the demand schedule, i.e., the bidding strategy, captures (inefficient) firms’ incentives to distort the demand that they propose to serve, and in this case, government payments recover distortion profits. The latter explains why the demand pricing mechanism dominates the naïve scoring auction, which does not account for distortions.

---

\(^{16}\) Except for the predetermined mechanism, the mechanisms self-adjust, which means that optimal schedules prescribe tolls and capacities that are increasing and decreasing in \( c_i \). Recall also that optimal schedules are fixed and independent of \( I \). The exception is that \( I \) affects the budget-balance constraint, which means that the optimal toll and capacity schedules in BBMD do depend on \( I \).
Finally, we present numerical examples to compare the aforementioned mechanisms/auctions, and highlight aspects of the above discussion. We include an auction with predetermined toll and capacity levels as a reference. Important observations and insights are that:

- The virtual surplus is an auxiliary function that corresponds to the conditional contribution to the ex-ante public welfare, given the winning firm’s variable cost, \( c_w \). The relative performance of the mechanisms is determined by the virtual surplus that is generated;
- The ex-ante public welfare is a weighted sum of the virtual surplus functions. The weights derive from the probability density function for the set of variable costs, \( \phi(\cdot) \), which, in turn, depends on the number of firms participating in the auction. We observe that, with little competition, there are clear differences among the mechanisms, but as competition increases, the performance of the naive, and demand pricing mechanisms converges to that of the optimal mechanism. This is because an efficient firm, with little incentive to distort, is more likely to win the concession, and the information rent is more likely to be small.

The numerical example also hints at limitations of the analysis herein, where we use the expected public welfare as a governments criterion. In particular, we show that in certain scenarios, i.e., for certain sets of realized bids, though less probable, the ranking of the mechanisms can be significantly different. This is common in decision-making under uncertainty, where the adage “good decisions can lead to bad outcomes” applies. This has motivated analysis that does not rely on a specification of a probability distribution function of types that is common knowledge for all players. Instead, and to ensure tractability, prior-free mechanisms impose a stronger version of IC, where truth-telling is the dominant strategy for any combination of types (Vickrey, 1961). As discussed in Hartline (2020), another approach to deal with uncertainty in specifying a type distribution is to apply the optimal mechanism for a worst-case scenario.

To conclude, we note that there are a number of directions to extend the present analysis. Examples include accounting for systematic or random demand fluctuations, or considering the design of combinatorial auctions to award concessions for links within road networks. As discussed in Engel et al. (1997); de Rus and Nombela (2000) and reviewed herein, the issue of demand uncertainty is relevant and has been analyzed in the context of auctions for road concessions (with predetermined toll and capacity levels). Extending the analysis to allow for uncertain or dynamic demand would, therefore, address an important limitation of our work. As shown in Strausz (2006); Pavan et al. (2014) and elsewhere in the economics literature, such extensions are non-trivial. Moreover, at this point, they would detract from the focus of the current paper on the effect of imperfect information. For example, it may be hard to separate distortions stemming from information asymmetries from optimal decisions trading off the relative costs of (capacity) overages or underages.

**CRediT authorship contribution statement**

Hang Shu: Conceptualization, Methodology, Formal analysis, Writing - original draft. Pablo L. Durango-Cohen: Conceptualization, Methodology, Formal analysis, Writing - original draft, Supervision.

**Acknowledgments**

The authors gratefully acknowledge funding provided by the Northwestern University Transportation Center through a Dissertation Year Fellowship awarded to Hang Shu.

**Appendix A. Proofs and simplifications**

**Proposition 1  Outcomes of predetermined toll-capacity bidding**

1. Bidder \( i \)'s interim surplus, i.e., the conditional expected profit, is

\[
\int_{\pi_{\min}}^{v_i} H^{-1}(y)dy = K^*(\tilde{c}) \int_{c_i}^{c_{\max}} F^{-1}(x)dx
\]

where \( H(\cdot) \) represents the cumulative distribution function corresponding to density \( h(\cdot) \), and \( v_i = \pi(\cdot; \tau^*(\tilde{c}), N^*(\tilde{c}), K^*(\tilde{c})) \) denotes firm \( i \)'s ex-post profit.

2. The ex-ante public welfare is

\[
E_w \left[ CS(\tau^*(\tilde{c}), N^*(\tilde{c}), K^*(\tilde{c})) + \pi (w; \tau^*(\tilde{c}), N^*(\tilde{c}), K^*(\tilde{c})) - \frac{F(w)}{f(w)} \cdot K^*(\tilde{c}) \right]
\]

where \( E_w \) denotes the expectation over the winning firm’s variable cost, \( w \).
Proof.

1. Following Riley and Samuelson (1981), firm \( i \)'s reservation value, \( v_i = \pi (c_i; \tau (\tilde{c}), N (\tilde{c}), K (\tilde{c})) \). Under equilibrium bidding strategies for first-price auctions, a firm with reservation value \( v_i \) receives an (interim) expected payment, \( M (v_i) \), given by

\[
M (v_i) = \int_{v_i}^{\pi_{\min}} H^{i-1} (y) dy - v_i \cdot H^{i-1} (v_i)
\]

Notice \( M (v_i) \) is not the bidding price, but the expected payment in advance of the auction, i.e., before the private bidder knows if it is the winner. Thus, the **interim** private surplus is given by

\[
M_i (v_i) + v_i \cdot P (\text{winning with } v_i) = \int_{v_i}^{\pi_{\min}} H^{i-1} (y) dy - v_i \cdot H^{i-1} (v_i) + v_i \cdot H^{i-1} (v_i)
\]

The last expression is from a variable change from profit to variable cost, including noting that, from (6), \( dy = -K^* (\tilde{c}) dx \).

2. Thus, the **ex-ante** private surplus for a random bidder is:

\[
\int_{\pi_{\min}}^{\pi_{\max}} \left[ \int_{\pi_{\min}}^{\pi_{\max}} H^{i-1} (y) dy \right] h (w) dw = \int_{\pi_{\min}}^{\pi_{\max}} \left[ \int_{y}^{\pi_{\max}} h (w) dw \right] H^{i-1} (y) dy
\]

\[
= \int_{\pi_{\min}}^{\pi_{\max}} \left[ 1 - H (y) \right] H^{i-1} (y) dy
\]

\[
= \int_{\pi_{\min}}^{\pi_{\max}} \frac{1 - H (y)}{I \cdot h (y)} \cdot H^{i-1} (y) \cdot h (y) dy
\]

where the first step comes from changing the integration order. Because that the cumulative probability distribution of the winner’s type \( H \). Thus, the corresponding density function is \( \frac{d H (y)}{dy} = I \cdot H^{i-1} (y) \cdot h (y) \), which is in the integrand. This, means that each of the bidders’ contribution to the private surplus is

\[
E_p \left[ \frac{1 - H (y)}{I \cdot h (y)} \right] = \frac{1}{I} E_p \left[ \frac{1 - H (y)}{h (y)} \right]
\]

where \( E_p \) is the expectation following the winner’s profit distribution. Since there are \( I \) bidders and the public welfare is the social welfare minus private surplus, the **ex-ante** public welfare is given by

\[
E_p \left[ SW (y; \tilde{c}) - \frac{1 - H (y)}{h (y)} \right] = E_p \left[ CS (\tilde{c}) + \pi (x; \tilde{c}) - \frac{F (x)}{f (x)} \cdot K^* (\tilde{c}) \right]
\]

where \( E_p \) is the expectation following the winner’s unit cost distribution.

□

**Proposition 2.** Generalized Monotonicity Condition

(Q, M, T, L) satisfies the Bayesian IC Condition if and only if

1. \( S_i (c_i) \equiv G_i (c_i, \cdot) \), i.e., \( S_i (c_i) \) is the value of \( G_i (\cdot) \) when firm \( i \) claims \( c_i \), i.e., \( \tilde{c_i} = c_i \).

\[
V_i (c_i) = V_i (\tilde{c_i}) + \int_{\tilde{c_i}}^{c_i} S_i (x) dx \quad \forall c_i \in C, i = 1, \ldots, I
\]

and

2. \( G_i (\cdot) \) satisfies the Generalized Monotonicity Condition given by

\[
\int_{\tilde{c_i}}^{c_i} (G_i (c_i, x) - G_i (\tau, x)) dx \geq 0, \quad \forall c_i \in C, i = 1, \ldots, I
\]

**Proof.**

1. They Bayesian Incentive Compatibility Condition implies that \( c_i = \arg \max_{c_i \in C} U_i (c_i, c_i) \). From the Envelope Theorem, it follows that

\[
\frac{\partial V_i (c_i)}{\partial c_i} = \frac{\partial U_i (c_i, c_i)}{\partial c_i} = S_i (c_i) \iff
\]
\[ V_i(c_i) - V_i(c'_i) = \int_{c'_i}^{c_i} S_i(x) \, dx \]

\[ U_i(c_i, c'_i) - U_i(c'_i, c'_i) \geq U_i(c_i, c'_i) - U_i(c, c) \Rightarrow \]
\[ \int_{c'_i}^{c_i} G_i(c_i, x) \, dx \geq \int_{c'_i}^{c_i} S_i(x) \, dx \Rightarrow \]
\[ \int_{c'_i}^{c_i} (G_i(c_i, x) - G_i(x, x)) \, dx \geq 0 \]

which shows that the Generalized Monotonicity Condition is necessary.

2. From the Generalized Monotonicity Condition and Definition 2.2 we have that
\[ \int_{c_i}^{c'_i} (G_i(c'_i, x) - G_i(x, x)) \, dx \geq 0 \Rightarrow \]
\[ U_i(c'_i, c_i) - U_i(c'_i, c'_i) + U_i(c, c'_i) \geq 0 \Rightarrow \]
\[ U_i(c_i, c'_i) \geq U_i(c'_i, c'_i) \]

which shows that the Generalized Monotonicity Condition is sufficient.

Thus, the conditions are equivalent. \( \square \)

Simplification of problem MD
The expected public welfare can be rewritten as follows:
\[
E_c \left[ \sum_{i=1}^{l} \left( q_i(c) \cdot CS(T(c), N(c), L(c)) \right) - M_i(c) \right] 
\]
\[
= \sum_{i=1}^{l} \left[ E_c[q_i(c) \cdot CS(T(c), N(c), L(c))] \right] - E_c \left[ E_{c_i} \left[ M_i(c_i, c) \right] \right] 
\]
\[
= \sum_{i=1}^{l} \left[ E_c[q_i(c) \cdot CS(T(c), N(c), L(c))] + E_c \left[ \pi_i(c_i) \cdot \pi(c_i; c_i; T(c), N(c), L(c)) \right] - V_i(c)) \right] 
\]
\[
= \sum_{i=1}^{l} \left[ E_c[q_i(c) \cdot (CS(T(c), N(c), L(c))) + \pi(c_i; T(c), N(c), L(c))) \right] - E_c[V_i(c))] 
\]

where the second expression comes from the definition of indirect utility in (11), \( V_i(c_i) \), and the third expression comes from the definition of conditional expectation \( \pi_i(c_i) \). We can also use the Envelope Theorem and apply integration by parts to obtain an expression for the last set of terms in the above expression:

\[
E_{c_i}[V_i(c_i)] = \int_{c_{min}}^{c_{max}} V_i(x) f(x) \, dx 
\]
\[
= F(x)V_i(x) \bigg|_{c_{max}}^{c_{min}} - \int_{c_{min}}^{c_{max}} (-\pi_i(x) \cdot L(x)) F(x) \, dx 
\]
\[
= V_i(c_{max}) + \int_{c_{min}}^{c_{max}} (\pi_i(x) \cdot L(x)) F(x) \, dx 
\]
\[
= V_i(c_{max}) + \int_{c_{min}}^{c_{max}} \left( \pi_i(x) \cdot L(x) \cdot \frac{F(x)}{F(c_i)} \right) f(x) \, dx 
\]
\[
= V_i(c_{max}) + E_c \left[ q_i(c) \cdot L(c_i) \frac{F(c_i)}{F(c_i)} \right] 
\]

Thus, the expected public welfare is
\[
\sum_{i=1}^{l} E_c \left[ q_i(c) \cdot \left( CS(T(c), N(c), L(c)) + \pi(c; T(c), N(c), L(c)) - L(c) \frac{F(c)}{F(c_i)} \right) \right] - V_i(c_{max}) 
\]
\[
= E_c \left[ \sum_{i=1}^{l} q_i(c) \cdot \left( CS(T(c), N(c), L(c)) + \pi(c; T(c), N(c), L(c)) - L(c) \frac{F(c_i)}{F(c_i)} \right) \right] - V_i(c_{max}) 
\]

**Proposition 3** Optimal budget-balanced mechanism
Under a regular, common prior, the following direct mechanism optimizes BBMD:
1. $Q^B$ is given by $q_i(c) = \begin{cases} 1 & : c_i = \min \{c_1, c_2, \ldots, c_I\}, \\ 0 & : \text{otherwise} \end{cases}$.

2. Transfer is given by,

$$M^B(c) = \begin{cases} -q_i(c) \cdot [\pi(c_1; T^B(c_1), N^B(c_1), L^B(c_1)) + \frac{\int_{c_{\min}}^{c_{\max}} P_w(x)L^B(x)dx}{P_w(c_w)}] & : P_1(c_i) \neq 0, \\ 0 & : \text{otherwise} \end{cases}$$

3. For each firm $i$, $T^B(c_i)$ and $L^B(c_i)$ solve the following Ramsey Problem $\text{RP-R}$:

$$\text{RP - R} \max_{\tau, K} \int_0^N P(n)dn - N \cdot C(N, K) - C'(\gamma_1(c_i); K)$$

subject to:

$$P(N) = \tau + C(N, K)$$

$$P_1(c_i) \cdot (N \cdot \tau - C'(c_i; K)) \geq -\int_{c_{\min}}^{c_{\max}} P_1(x)L^B(x)dx$$

**Proof.** We want to show that the mechanism $(Q^B, L^B, T^B, M^B)$ as presented in the proposition is the optimal solution to problem $BBMD$. Following a similar logic to the one used in analyzing MD-S, we can verify that the mechanism is IC, and ex-post IR, i.e., that constraints (36), (37), and (39) are satisfied. To verify that the mechanism is budget balanced, we note that the ex-post total payment is given by

$$\sum_{i=1}^l M^B_i(c) = -\pi(c_w; T^B(c_w), N^B(c_w), L^B(c_w)) - \int_{c_{\min}}^{c_{\max}} P_w(x)L^B(x)dx \leq 0$$

where the inequality follows from the second constraint in $\text{RP-R}$ and $w$ is the winner. Thus, the mechanism is a feasible solution to $BBMD$.

We proceed by contradiction to verify that the mechanism is optimal. Assuming there is another IC, ex-post IR, and ex-post budget balanced mechanism, $(Q', T', L', M')$, yielding a strictly higher ex-ante public welfare than $(Q^B, T^B, L^B, M^B)$. Recall that the ex-ante public welfare is the expectation of the virtual surplus over the bidders. Thus, there exists at least one firm, $j$, with variable cost, $c_j$, such that $R(c_j, T', L') > R(c_j, T^B, L^B)$. Noticing that the objective function in $\text{RP-R}$ is the virtual surplus, it must be the case that the profitability constraint in $\text{RP-R}$ is violated. Then,$^{17}$ Thus,

$$P_1(c_j) \cdot \pi(c_j; T'(c_j), N'(c_j), L'(c_j)) < -\int_{c_{\min}}^{c_{\max}} P_j(x)L'(x)dx$$

Now, from the IC constraint

$$E_{c_{\min}}[M_j^B(c)] = -P_j(c_j) \cdot \pi(c_j; T'(c_j), N'(c_j), L'(c_j)) - \int_{c_{\min}}^{c_{\max}} P_j(x)L'(x)dx > 0$$

When the conditional expectation of the payment is positive, there must be a positive ex-post payment, for some $\tilde{c}_{-w}$. Due to the ex-post IR, losing firms don’t pay, i.e., $M'_j(c) \geq 0 \ \forall i \neq w$. Thus, $j$ must be the winning firm, and the total payment

$$\sum_{i=1}^l M^B_i(c) \geq M'_j(c) > 0,$$

which contradicts to the ex-post budget balance constraint. □

**Lemma 1** Optimal toll and capacity levels in Scoring Auctions

With a scoring function $a(\tau, K, M) = \alpha(\tau, K) - M$, firm $i$ chooses toll and capacity levels, $\tau$ and $K$, that maximize $\alpha(\tau, K) + \pi(c_i; \tau, N(\tau, K), K)$.

**Proof.** Suppose firm $i$ selects the prescribed $\tau, K, M$ that maximize

$$\{\pi(c_i; \tau, N(\tau, K), K) + M\} \cdot P(\text{winning with } a(\tau, K, M))$$

, but there is another set, $\tau', K'$, such that, $\alpha(\tau', K') + \pi(c_i; \tau', N(\tau', K'), K') > \alpha(\tau, K) + \pi(c_i; \tau, N(\tau, K), K)$. Letting $M' = \alpha(\tau, K') - M - \alpha(\tau, K)$, we have $a(\tau, K, M) = \alpha(\tau, K) - M = \alpha(\tau', K') - M' = a(\tau', K', M')$, i.e., the bid with $\tau', K', M'$ yields the same score and probability of winning as that of firm $i$’s preferred bid. However, it also yields a total profit that is higher than the equilibrium bid as

$$\pi(c_i; \tau', N(\tau', K'), K') + M' = \pi(c_i; \tau', N(\tau', K'), K') + \alpha(\tau', K') + M - \alpha(\tau, K)$$

$$> \pi(c_i; \tau, N(\tau, K), K) + a(\tau, K, M) + M - \alpha(\tau, K, M)$$

$$= \pi(c_i; \tau, N(\tau, K), K) + M$$

$^{17}$ User equilibrium condition has to be satisfied in any feasible mechanism.
contradicting to the optimality of the bid with \( \tau, K, M \). □

**Proposition 4** Optimal Scoring Function

A first-best scoring auction with the scoring function appearing below results in an implementation of the Optimal Direct Mechanism.

\[
\alpha^d(\tau, K) = \text{CS}(\tau, N(\tau, K), K) - \int_0^K \frac{F(c^p(k))}{f(c^p(k))} dk
\]

where \( c^p(\cdot) \) is the inverse of the public optimal capacity schedule, \( L^c(c_i) \). Furthermore, adding the constraint \( M \leq 0 \) and adjusting the inverse schedule result in an implementation of the Budget-Balanced Public-Optimal Mechanism.

**Proof.** From Lemma 1, each firm maximizes \( \alpha^d(\tau, K) + \pi(c_i; \tau, N(\tau, K), K) = \text{SW}(N(\tau, K), K) - \int_0^K \frac{F(c^p(k))}{f(c^p(k))} dk \), which corresponds to an adjusted social welfare function with cost function \( \tilde{C}(c_i, K) = c_0 + c_i \cdot K + \int_0^K \frac{F(c^p(k))}{f(c^p(k))} dk \). From the Regularity Assumption, i.e., Assumption 4, \( \tilde{C}(\cdot) \) is a convex function in \( \tau \) and \( K \). Along with Assumption 1, we have that a point satisfying the conditions below results in an optimal bid.

\[
\frac{\partial \text{SW}(\cdot)}{\partial \tau} = 0 \quad \text{and} \quad \frac{\partial \text{SW}(\cdot)}{\partial K} - \frac{F(c^p(K))}{f(c^p(K))} = 0
\]

which we note are the same conditions for a point optimizing virtual surplus. We can turn to the constrained optimization problem by claiming \( M \leq 0 \) beforehand, and apply the similar trick to implement the budget-balanced version of the optimal mechanism. □

**Lemma 3** Regularity Condition for Demand Pricing Mechanism

For any congestion function, \( g(\cdot) \). \( N^d \) is decreasing with respect to \( c_i \) if and only if

\[
2c_i g'(c_i) > \frac{F(c_i)}{f(c_i)}, \quad c_i \in C, \quad i = 1, \ldots, I
\]

**Proof.** From Eq. (52), \( N^d \) strictly decreases, if and only if for any \( c_i \), \( g'(c_i \mu^* - \frac{F}{f} \mu^* \varepsilon > 0 \). Differentiating (5) with respect to \( c_i \) gives

\[
g'' \mu^2 \varepsilon + 2g' \mu^* \varepsilon = 1
\]

Noticing that \( \mu^* > 0 \), \( \mu^* (c_i \varepsilon > 0 \), and that \( g'' \geq 0 \), we have

\[
2g' \mu^* \varepsilon \leq 1 \Rightarrow 2g' \mu^2 \varepsilon \leq \mu^* \Rightarrow 2c_i \leq \frac{\mu^*}{\mu^2}
\]

where the last line comes from (5). Now, we revisit the conditions on the derivatives,

\[
\frac{\gamma_i}{c_i} g' \mu^* - \frac{F}{f} \mu^* \varepsilon > 0 \Rightarrow \\
\frac{\gamma_i}{c_i} g' \mu^* > \frac{F}{f} \mu^* \varepsilon > 0
\]

In terms of the arguments for the proof:

Sufficiency: If \( 2c_i g' > \frac{F}{f} \mu^* \varepsilon \), then \( \gamma_i \mu^* > \frac{F}{f} \mu^* \varepsilon - \frac{F}{f} \frac{\mu^*}{\mu^2} > 0 \).

Necessity: When \( g'' = 0 \), (A.1) becomes equality. Then, \( 2c_i g' \mu^* \varepsilon \) is equivalent to \( \gamma_i \mu^* \varepsilon \mu^* \varepsilon \varepsilon \) and the inequality, \( 2c_i g' > \frac{F}{f} \mu^* \varepsilon \), has to be satisfied. □

**Proposition 5** \( N^d(c_i) \) decreasing with respect to \( c_i \) ensures that the monotonicity condition in \( \text{DP} \), Eq. (48), is satisfied.

**Proof.** Assumption 4 implies the probability of winning, \( \mathbb{P}(c_i) \), decreasing with \( c_i \). The assumption that \( N^d(c_i) \) is decreasing with respect to \( c_i \) means that \( K^d(c_i) = \frac{N^d(c_i)}{\mathbb{P}(c_i)} \) is also decreasing with \( c_i \) because \( \mu^* > 0 \) is increasing with \( c_i \). Thus, \( S_i(c_i) = -\mathbb{P}(c_i) \cdot K^d(c_i) \) is non-decreasing in \( c_i \). □

**Proposition 6** The government implements the optimal Demand-Pricing Mechanism, \( \text{DP} \), by offering \( Y \), as specified from Eq. (53), and selecting the proposal with the highest demand.

**Proof.** From the Revelation Principle, the demand pricing is optimal as long as, bidder with each \( c_i \) always selects the optimal demand \( N^d(c_i) \) and the government selects the firm with the lowest \( c_i \). Assume the bidder \( i \) selects a pair \( (N', Y(N')) \), different from the optimal revealed decision, \( (N^d(c_i), Y(N^d(c_i))) \). Then, due to unique mapping, \( (N', Y(N')) \) must correspond to another variable cost, \( (N^d(c_i), Y(N^d(c_i))) \). Due to the IC constraints, the expected utility from \( (N', Y(N')) \) is lower than that from \( (N^d(c_i), Y(N^d(c_i))) \). Thus, each bidder \( i \) will bid the optimal demand corresponding to their true variable cost.
\[ N^d(c_i) \text{. Next, as } N^d \text{ is strictly decreasing, the highest demand corresponds to the lowest variable cost. Thus, the bidder bidding the highest demand is the most efficient firm. Putting both parts together, awarding the franchise to the bid with the highest demand, and offering } Y \text{ yield an implementation of the Optimal Demand Pricing Mechanism.} \]

\begin{lemma}
\textbf{Performance Bound}
\end{lemma}

The virtual surplus loss from implementation of the Demand Pricing Mechanism is always less than the loss stemming from implementation of the commonly-used Naive Scoring Rule, i.e., \( \Delta_n - \Delta_d \geq 0 \).

**Proof.** From the definitions,

\[
\Delta_n - \Delta_d = \frac{F(c_i)}{f(c_i)} K^*(c_i) - \int_0^\gamma K^*(h) \, dh - \int_0^\gamma (\gamma_i - h) N^*(h, \gamma_i) - \frac{\mu^*(h)}{\mu^*(\gamma_i)} \, dh
\geq \frac{F(c_i)}{f(c_i)} K^*(c_i) - \int_0^\gamma N^*(h, \gamma_i) - \frac{\mu^*(h)}{\mu^*(\gamma_i)} \, dh
\geq \frac{F(c_i)}{f(c_i)} K^*(c_i) - N^*(c_i, c_i) \cdot \left[ \int_0^\gamma \frac{\gamma_i \cdot \mu^*(h)}{\mu^*(\gamma_i)} \, dh + \int_0^\gamma \frac{\mu^*(h) - h \cdot \mu^*(h)}{\mu^*(\gamma_i)} \, dh \right]
\geq \frac{F(c_i)}{f(c_i)} K^*(c_i) - N^*(c_i, c_i) \cdot \left[ \frac{\gamma_i}{\mu^*(\gamma_i)} - \frac{h}{\mu^*(h)} \right]_0^\gamma + \frac{\gamma_i}{\mu^*(\gamma_i)} - \frac{1}{\mu^*(\gamma_i)} - c_i
\geq \frac{F(c_i)}{f(c_i)} K^*(c_i) - N^*(c_i, c_i) \cdot \frac{\gamma_i}{\mu^*(\gamma_i)} = \frac{F(c_i)}{f(c_i)} K^*(c_i) - N^*(c_i, c_i) \frac{1}{\mu^*} = 0
\]

\[\square\]

**References**


Hartline, J.D. 2020. Mechanism Design and Approximation. Northwestern University, Evanston, IL.


University Library of Munich, Germany.


