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Community Enforcement with Endogenous Records

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I study repeated games with anonymous random matching where players can add or remove signals from their records. The ability to manipulate records introduces monotonicity constraints on players' continuation values, under which sufficiently long-lived players will almost never cooperate. When players' expected lifespans are intermediate, their ability to sustain cooperation depends on (i) whether their actions are complements or substitutes and (ii) whether manipulation takes the form of adding or removing signals.

Key words: Community enforcement, Endogenous records, Expected lifespan, Anti-folk theorem

JEL codes: C73, D82, D83

1. INTRODUCTION

When will a group of selfish individuals cooperate with one another? This classic question has motivated the game-theoretic literature on *community enforcement*. In small communities with relatively few players, Kandori (1992), Ellison (1994), and Deb *et al*. (2020) show that players can cooperate even when they have no information about others' histories. In *large* communities with many players, which are usually modelled as a continuum, Takahashi (2010) shows that sustaining cooperation requires players to have some information about their partners' histories. Such information is called a player's *record*, which may consist of signals about his past actions and possibly also signals about his previous partners' actions.

This paper studies community enforcement in large communities where players' records are *endogenous* in the sense that they can add or remove signals from their records. One domain of applications is online reviews in which firms may persuade consumers to erase negative reviews or to write positive ones. My analysis implies that (i) the maximal amount of cooperation a community can sustain is not monotone with respect to its members' expected lifespans, and (ii) whether the complementarity of players' actions is conducive to cooperation depends on whether manipulation takes the form of adding or removing signals.

To provide an overview of my model and results, consider a simple example with a continuum of players. In each period, all the active players are randomly matched into pairs to play the prisoner's dilemma. Each player's action generates a *signal*, and his *record* consists

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of a sequence of signals. At the end of each period, a fixed fraction of players irreversibly exit the game (*i.e.* become inactive), replaced by the same mass of new players. This exit process introduces an additional source of discounting, alongside players' time preferences. In the next period, the remaining active players are matched with new partners.

My modelling innovation is that *before* players are matched with new partners, they can manipulate their records subject to *feasibility constraints*. I consider two forms of manipulation, which correspond to two classes of feasibility constraints. I say that a player can *erase* signals if he can choose his record to be any subsequence of the signals generated by his past actions. I say that a player can *add* signals if he can choose any record such that the sequence of signals generated by his past actions is a subsequence of that record. I assume that their new partners can only observe their *manipulated records* but not their age in the game.

My main result, Theorem 1, shows that *sufficiently long-lived* players will almost never cooperate in any equilibrium (i) if they can erase signals *or* (ii) if they can add signals and their signals are noisy.

The intuition behind Theorem 1 is that players' ability to erase or add signals introduces *monotonicity constraints* on their continuation values. In the case where a player can *erase* signals, he can always replicate his current record in the next period. This implies that in any equilibrium, his continuation value must be *non-decreasing* over time. As a result, he has an incentive to cooperate only if doing so can significantly increase his continuation value, so the expected number of periods in which he cooperates must be bounded above. When this player has a sufficiently long expected lifespan, any bounded number of periods carry negligible weight, so the average probability with which he cooperates must be close to zero.

When a player can *add* signals, his continuation value is *non-increasing* over time since he can always replicate his future records in the current period. As a result, he has an incentive to cooperate *only if* his continuation value after he defects is significantly lower than his current continuation value. Assuming that the signals that monitor his actions are noisy, his continuation value after he cooperates must also be lower than his current continuation value. Hence, the expected number of periods in which the player has an incentive to cooperate is bounded above. By the same logic as in the case of erasing signals, when a player's expected lifespan diverges to infinity, the average probability with which he cooperates vanishes to zero.

Theorem 1 suggests that sufficiently long-lived players will almost never cooperate. Sufficiently short-lived players have no incentive to cooperate since their discount factors are too low. A natural question is: Can players sustain *some* cooperation in *some* equilibria when their expected lifespans are *intermediate*?[1]

Theorem 2 shows that as long as players are not too impatient, have *intermediate* expected lifespans, and the signals that monitor their actions are precise enough, they can sustain some cooperation *either* when they can only erase signals and have submodular payoffs *or* when they can only add signals and have supermodular payoffs. This together with Theorem 1 suggests that the maximal level of cooperation is not monotone with respect to players' expected lifespans.[2] The cooperative equilibria I construct are *purifiable*, which means that they are robust when players have a small amount of private payoff information.

1. There is always an equilibrium where players always defect, which rules out sustaining cooperation in all equilibria. I also show that the average probability of cooperation is *uniformly* bounded below 1 in *all* equilibria under *all* time preferences and expected lifespans (or equivalently, survival probabilities). This rules out the possibility of sustaining full cooperation.

2. Wiseman (2017) and Sandroni and Urgun (2018) also show that higher effective discount factors can undermine cooperation. In contrast to the current paper that focuses on repeated games, their results are obtained in stochastic games with absorbing states.

Theorem 3 shows that regardless of their time preferences and survival probabilities, players will always defect in all purifiable equilibria *either* when they can erase signals and have supermodular payoffs *or* when monitoring is noisy, they can add signals, and have submodular payoffs. My result implies that when players can only erase signals, the complementarity of their actions *undermines* their abilities to sustain cooperation. This conclusion stands in contrast to the ones in Takahashi (2010), Heller and Mohlin (2018), and Clark *et al*. (2021), which suggest that the complementarity of players' actions is conducive to cooperation in community enforcement models where players can neither add nor erase signals.

This paper is related to the existing works on community enforcement, and in particular, those that analyse games with a continuum of players such as Takahashi (2010), Heller and Mohlin (2018), Bhaskar and Thomas (2019), and Clark *et al*. (2021). Friedman and Resnick (2001) study repeated prisoner's dilemma in large populations where each player can *either* disclose all of his past signals *or* erase all of them. In contrast, the players in my model can decide whether and when to erase or to add each signal.

Ali and Miller (2016) study repeated games with a finite number of players where players can selectively disclose *the actions of their previous partners* to their current partners. Due to players' incentives to conceal past deviations, equilibria that forgive past deviators can sustain more cooperation than those with permanent ostracism. My model of erasing signals can be interpreted as players selectively disclosing *signals about their past actions* to their current partners, which contrasts to the setting studied by Ali and Miller (2016).[3]

Smirnov and Starkov (2022), Hauser (2023), and Sun (2024) study dynamic censoring games where players' payoffs depend on an exogenous state. In contrast, players' payoffs depend only on their actions in my model. Pei (2023) studies a repeated game with incomplete information in which a long-lived player can erase past actions from his record. That paper presents a bad reputation result, driven by the observation that the speed of learning vanishes as the expected lifespan of the long-lived player diverges to infinity. By contrast, the current model has complete information so the speed of learning is irrelevant.

Ghosh and Ray (1996) and Fujiwara-Greve and Okuno-Fujiwara (2009) study repeated games with voluntary separation where players may interact with the same partner in multiple periods. In their models, each player's outside option is his continuation value from separation and joining the unmatched pool. This feature contrasts to my model where each player's outside option is his current or future continuation value.

2. THE BASELINE MODEL

I introduce a framework that allows for asymmetric stage games as well as erasing and adding signals. Consider a doubly infinite repeated game where time is indexed by $k = \cdots - 1, 0, 1, \ldots$. There are two populations of players $i \in I \equiv \{1, 2\}$. Each period, a unit mass of players from each population are active.

Each player discounts future payoffs for two reasons. First, by the end of each period, a fraction $1 - \overline{\delta}_i$ of the active players in population i *irreversibly become inactive* and are replaced by the same mass of new players, with $\overline{\delta}_i \in [0, 1)$. Second, conditional on remaining active in period $k + 1$, each player in population i is indifferent between 1 unit of utility in period $k + 1$

3. Sugaya and Wolitzky (2020) establish an anti-folk theorem when there is a *finite* number of players, each player has private information about his type (rational or committed), and players face uncertainty regarding the composition of types in the population.

and $\widehat{\delta_i} \in [0,1)$ unit in period k. Hence, each player in population i has an *expected lifespan* $(1-\overline{\delta}_i)^{-1}$ and an *effective discount factor* $\delta_i \equiv \widehat{\delta_i} \cdot \overline{\delta}_i$.[4]

Each period, all the active players are matched into *pairs* uniformly at random to play a two-player normal form game $\mathcal{G} \equiv (I, A, u)$, where $A \equiv A_1 \times A_2$ is the set of action profiles with A_i a *finite* set of actions for players from population i (which I refer to as *player i*) and $u_i : A \to \mathbb{R}$ is player i's stage-game payoff. Each player in population i maximizes the expected value of $\sum_{k=1}^{+\infty}(1-\delta_i)\delta_i^{k-1}u_i(a_{i,k}, a_{-i,k})$ where $(a_{i,k}, a_{-i,k}) \in A_i \times A_{-i}$ stands for the action profile played by his match in the kth period of his life.

For every match, players' actions generate signals (s_1, s_2) according to $f(\cdot|a_i, a_{-i}) \in \Delta(S_1 \times S_2)$, where s_i is *player i's signal* with distribution $f_i(\cdot|a_i, a_{-i}) \in \Delta(S_i)$. I assume that S_1 and S_2 are finite sets.

Player i's *record* consists of a sequence of elements in S_i. Let $R_i \equiv \bigcup_{n=0}^{+\infty} S_i^n$ denote the set of player i's records, with a typical element denoted by $r_i \in R_i$. By definition, the empty record $\varnothing$ belongs to R_i.

My modelling innovation is that *before* each player is matched with a new partner, he may manipulate his record by erasing or adding signals. I discuss alternative forms of endogenous records in Section 5.

I say that player i can *erase* signals if before being matched in period $k+1$, he can erase the signal $s_{i,k} \in S_i$ generated by his pair in period k as well as any signal that belongs to his period-k record $r_{i,k}$. Formally, this is to say that player i can choose *any subsequence* of $(r_{i,k}, s_{i,k})$ to be his period-$(k+1)$ record. If player i can *only* erase signals, then before interacting with his first partner, his record must be $\varnothing$.

I say that player i can *add* signals if, before being matched with any partner, he can add any finite number of elements in S_i to his record, in addition to that generated by his past match. In this case, a newly born player i can choose his record from R_i before being matched with his first partner. If $r_{i,k}$ is a player i's record in period k and $s_{i,k} \in S_i$ is the signal generated by his pair's action profile in period k, then player i can choose his period-$(k+1)$ record to be any $r_{i,k+1} \in R_i$ such that $(r_{i,k}, s_{i,k})$ is a subsequence of $r_{i,k+1}$.[5]

After each player i is matched, he observes his record r_i and his partner's record r_{-i} (the one *after* manipulation) before choosing a_i. Players *cannot* directly observe any additional information about their partners, such as their partners' age in the game and which signals were erased or added by their partners. Players can make inferences about these variables via Bayes rule after observing their partners' records.

Player i's *strategy* is denoted by $\sigma_i \equiv (\sigma_i^{\varnothing}, \sigma_i^a, \sigma_i^m)$, where $\sigma_i^{\varnothing} \in \Delta(R_i)$ is his record choice before being matched with his first partner,[6] $\sigma_i^a : R_i \times R_{-i} \to \Delta(A_i)$ is a mapping from his current record r_i and his current partner's record r_{-i} to his current-period action, and $\sigma_i^m : R_i \times S_i \to \Delta(R_i)$ is a mapping from his current-period record r_i and his current-period signal s_i to the record his next partner observes.

4. I distinguish between players' time preferences and survival probabilities since they play different roles. This is reminiscent of the steady-state learning models in Fudenberg and Levine (1993), Fudenberg and He (2018), and Clark and Fudenberg (2021).

5. I comment on several extensions in Section 5, which include the newly added signals must come *after* $(r_{i,k}, s_{i,k})$ in the sequence of signals, players can only observe the summary statistics of others' signals but not the exact sequence, and so on.

6. As will become clear after I present the feasibility constraints, a player's record choice before being matched with his first partner is relevant *only if* he can add signals. My main result, Theorem 1, extends when players cannot add signals before being matched with his first partner since his first-period action carries negligible weight when his survival probability is close to 1.

Depending on whether and how player i can manipulate records, his choice of $(\sigma_i^{\varnothing}, \sigma_i^m)$ faces different feasibility constraints. When player i can neither erase nor add signals as in Clark *et al.* (2021), $\sigma_i^m(r_i, s_i)$ assigns probability 1 to (r_i, s_i). If player i can erase signals, then $\sigma_i^m(r_i, s_i)$ can assign positive probability to any subsequence of (r_i, s_i). If player i can add signals, then $\sigma_i^{\varnothing}$ can assign positive probability to any element in R_i and $\sigma_i^m(r_i, s_i)$ can assign positive probability to any $r_i' \in R_i$ such that (r_i, s_i) is a subsequence of r_i'. As long as player i *cannot* add signals, $\sigma_i^{\varnothing}$ must assign probability 1 to $\varnothing$.

The solution concept is steady-state Nash equilibrium, or *equilibrium* for short, which consists of a strategy profile $\sigma \equiv (\sigma_1, \sigma_2)$ and a record distribution $\mu \in \Delta(R_1 \times R_2)$ such that (i) for every $i \in \{1, 2\}$, σ_i maximizes the expected value of $\sum_{k=1}^{+\infty}(1-\delta_i)\delta_i^{k-1}u_i(a_{i,k}, a_{-i,k})$ when the record distribution is μ and players in the other population use strategy σ_{-i} and (ii) μ is a steady-state record distribution when players behave according to σ. An equilibrium exists in this repeated game since $\mathcal{G}$ is finite and players (i) always playing the same Nash equilibrium in $\mathcal{G}$ and (ii) never erasing or adding any signal is part of an equilibrium.

3. MAIN RESULT: ANTI-FOLK THEOREM WITH SUFFICIENTLY LONG-LIVED PLAYERS

Even though players can erase or add signals, they may still have incentives to cooperate (*i.e.* to take actions that are suboptimal in the stage game) when their effective discount factors are large enough. This is because: (i) in the case where players can *only* erase signals, they cannot fabricate good signals, so they may have incentives to cooperate if they are rewarded for having many good signals in their records; and (ii) in the case where players can *only* add signals, they cannot erase any bad signal generated by their actions, so they may have incentives to cooperate if they are punished for having many bad signals in their records.

My main result shows that the above logic breaks down when players are *sufficiently long-lived*. For any $\sigma \equiv (\sigma_1, \sigma_2)$ and μ, the *average probability* with which players in population i taking action $a_i \in A_i$, denoted by $\Pi_i^{(\sigma,\mu)}(a_i)$, is defined as the probability that $\sum_{(r_1,r_2)\in R_1\times R_2} \mu(r_1, r_2)\sigma_i^a(r_1, r_2)$ assigns to a_i.

Definition. *Player i's signal distribution f_i has non-shifting support if for every $a_{-i} \in A_{-i}$, $a_i, a_i' \in A_i$, and $s_i \in S_i$, we have $f_i(s_i|a_i, a_{-i}) > 0$ if and only if $f_i(s_i|a_i', a_{-i}) > 0$.*

My non-shifting support condition requires that the *support* of player i's signal distribution to be independent of his own action, which is to say that *monitoring is noisy*. It is weaker than f_i having *full support*, a condition commonly used in repeated games and reputations (*e.g.* (Cripps *et al.*, 2004)).

Theorem 1. *Suppose players in population i have a strictly dominant action $a_i^* \in A_i$.*

(1) *If players in population i can erase signals, then for every $\widehat{\delta}_i \in (0, 1)$ and $\varepsilon > 0$, there exists $\delta^* \in (0, 1)$ such that $\Pi_i^{(\sigma,\mu)}(a_i^*) \geq 1 - \varepsilon$ for every equilibrium (σ, μ) when $\overline{\delta}_i > \delta^*$.*
(2) *If population i can add signals and f_i has non-shifting support, then for every $\widehat{\delta}_i \in (0, 1)$ and $\varepsilon > 0$, there exists $\delta^* \in (0, 1)$ such that $\Pi_i^{(\sigma,\mu)}(a_i^*) \geq 1 - \varepsilon$ for every equilibrium (σ, μ) when $\overline{\delta}_i > \delta^*$.*[7]

7. To see why non-shifting support is not redundant in the case of adding signals, consider the prisoner's dilemma and suppose $S_i = A_i$ and s_i perfectly reveals a_i, *i.e.* $f_i(s_i = a_i|a_i, a_{-i}) = 1$ for every i and a_i. There is an equilibrium with full cooperation in grim-trigger strategies where each player cooperates if and only if no *defect* is contained in his and his partner's record.

Theorem 1 implies that as long as players are sufficiently long-lived and can *either* erase *or* add signals, they will almost always take their strictly dominant actions in all equilibria. In the prisoner's dilemma, it implies that sufficiently long-lived players will almost never cooperate. This result holds independently of the other population's stage-game payoffs, survival probabilities, time preferences, and whether and how they can manipulate their records. It applies for all finite action sets and signal spaces, even when these sets are sufficiently rich. It also applies when player i's effective discount factor δ_i is arbitrarily close to 1, such as when both $\widehat{\delta}_i$ and $\overline{\delta}_i$ go to 1 but $\overline{\delta}_i$ goes to 1 faster than $\widehat{\delta}_i$. This stands in contrast to the standard logic in repeated games, that fix player i's time preference $\widehat{\delta}_i$, an increase in his survival probability $\overline{\delta}_i$ leads to a higher effective discount factor δ_i, which will strengthen his incentive to sacrifice his current-period payoff in exchange for a higher payoff in the future and hence, may lead to more cooperation in equilibrium.

I present the proof of Theorem 1 in Section 3.1. The intuition is that players' ability to either erase or add signals introduces *monotonicity constraints* on their continuation values in the sense that their equilibrium continuation values must be either *non-decreasing* over time or *non-increasing* over time.

Suppose first that player i can *erase* signals. He can always replicate his current record in the next period, by taking an arbitrary action and then erase his signal s_i. This implies that in equilibrium, his continuation value must be *non-decreasing* over time. In order to motivate player i to take any action a_i' that is not his strictly dominant action a_i^*, his expected continuation value after playing a_i' needs to *increase* by at least something proportional to $1-\delta_i$. This implies that the expected number of periods in which player i taking actions other than a_i^* is no more than something proportional to $(1-\delta_i)^{-1}$. When players in population i are sufficiently long-lived (*i.e.* $\overline{\delta}_i \to 1$), the bounded number of periods in which they have incentives to cooperate will carry negligible weight, so the average probability that they cooperate must be close to 0.

Suppose next that player i can *add* signals. His continuation value is *non-increasing* over time since he can always replicate his future record in the current period. This implies that at every history where player i has an incentive to take action $a_i' \neq a_i^*$, his expected continuation value after playing a_i^* needs to *decrease* by at least something proportional to $1-\delta_i$ relative to his current continuation value. Assuming that f_i has non-shifting support, his expected continuation value after taking *any other action* also needs to decrease by at least something proportional to $1-\delta_i$. By the same logic as in the case of erasing signals, the expected number of periods in which player i has an incentive to cooperate is no more than something proportional to $(1-\delta_i)^{-1}$. As player i's expected lifespan diverges, the average probability of cooperation vanishes.

In Appendix A, I explain how to show Corollary 1, which is a result that applies to *all* stage games (including ones without dominant actions), by iteratively applying the above logic and by using the fact that the matching process is uniform. Let $A_i^* \subset A_i$ denote the set of player i's *rationalizable actions* in $\mathcal{G}$.

Corollary 1. *Suppose for every* $i \in \{1,2\}$*, either players in population i can erase signals, or they can add signals and* f_i *has non-shifting support. For every* $\widehat{\delta}_1, \widehat{\delta}_2 \in (0,1)$ *and* $\varepsilon > 0$*, there exists* $\delta^* \in (0,1)$ *such that when* $\overline{\delta}_1, \overline{\delta}_2 > \delta^*$, $\min_{i\in\{1,2\}} \sum_{a_i \in A_i^*} \Pi_i^{(\sigma,\mu)}(a_i) \geq 1-\varepsilon$ *in every equilibrium* (σ,μ).

3.1. *Proof of Theorem 1*

Fix an equilibrium (σ,μ). Recall that $\sigma_i^a(r_i, r_{-i}) \in \Delta(A_i)$ stands for player i's equilibrium action when his record is r_i and his matched partner's record is r_{-i}. Let $R_i^* \subset R_i$ denote the set of player i's records that occur with positive probability under μ. Let $V(r_i)$ denote player i's expected continuation value when his current-period record (*after* manipulation) is r_i *before* knowing his current match. Let $\overline{V} \equiv \sup_{r_i \in R_i^*} V(r_i)$ and $\underline{V} \equiv \inf_{r_i \in R_i^*} V(r_i)$ denote player i's highest and lowest continuation values, respectively. Conditional on being active for at least $k \in \mathbb{N}$ periods, let V_k denote player i's *expected* continuation value in the kth period of his life and let π_k denote his expected probability of taking actions that are *not* a_i^* in the kth period of his life. Let $\overline{u}_i$ and $\underline{u}_i$ denote player i's highest stage-game payoff and minmax value, respectively. Let

$$c^* \equiv \min_{a_i' \neq a_i^*, a_{-i} \in A_{-i}} \left\{ u_i(a_i^*, a_{-i}) - u_i(a_i', a_{-i}) \right\} > 0, \tag{1}$$

which is player i's lowest stage-game cost of taking actions other than a_i^*. I consider two cases separately.

First, suppose player i can erase signals. For every $r_i \in R_i^*$ and $s_i \in S_i$, let $R_i(r_i, s_i) \subset R_i^*$ denote the set of player i's records that are subsequences of (r_i, s_i). Since r_i is a subsequence of (r_i, s_i) and $r_i \in R_i^*$, we know that $R_i(r_i, s_i)$ is a non-empty finite set. Hence, player i's continuation value in the next period is $\max_{r_i' \in R_i(r_i, s_i)} V(r_i')$ when his current-period record is r_i and his current-period signal is s_i. We know from $r_i \in R_i(r_i, s_i)$ that $\max_{r_i' \in R_i(r_i, s_i)} V(r_i') \geq V(r_i)$. That is to say, player i's continuation value is *non-decreasing* over time, regardless of the realization of signal s_i. When player i with record r_i is matched with an opponent with record r_{-i}, he has an incentive to take action $a_i' \neq a_i^*$ only if

$$\begin{aligned}
&(1-\delta_i) u_i(a_i', \sigma_{-i}^a(r_i, r_{-i})) + \delta_i \sum_{s_i \in S_i} f_i(s_i | a_i', \sigma_{-i}^a(r_i, r_{-i})) \max_{r_i' \in R_i(r_i, s_i)} V(r_i') \\
&\quad \geq (1-\delta_i) u_i(a_i^*, \sigma_{-i}^a(r_i, r_{-i})) + \delta_i V(r_i),
\end{aligned}$$

where the RHS is player i's payoff when he plays a_i^* and then erases signal s_i regardless of its realization. The above inequality together with the definition of c^* implies that

$$\sum_{s_i \in S_i} f_i(s_i | a_i', \sigma_{-i}^a(r_i, r_{-i})) \max_{r_i' \in R_i(r_i, s_i)} V(r_i') \geq V(r_i) + \frac{1-\delta_i}{\delta_i} c^*,$$

which is to say that as long as player i has an incentive to play a_i' at (r_i, r_{-i}), his expected continuation value in the next period after playing a_i', which equals $\sum_{s_i \in S_i} f_i(s_i | a_i', \sigma_{-i}^a (r_i, r_{-i})) \max_{r_i' \in R_i(r_i, s_i)} V(r_i')$, is at least $\frac{1-\delta_i}{\delta_i} c^*$ greater than his continuation value at r_i. This leads to a lower bound on $V_{k+1} - V_k$:

$$V_{k+1} - V_k \geq \frac{1-\delta_i}{\delta_i} c^* \pi_k. \tag{2}$$

Since $\underline{u}_i \leq \underline{V} \leq V_k \leq \overline{V} \leq \overline{u}_i$ for every $k \in \mathbb{N}$, summing up (2) for all $k \in \mathbb{N}$, we have

$$\frac{1-\delta_i}{\delta_i} c^* \sum_{k=1}^{+\infty} \pi_k \leq \sum_{k=1}^{+\infty} (V_{k+1} - V_k) \leq \overline{V} - \underline{V} \leq \overline{u}_i - \underline{u}_i. \tag{3}$$

Since player i survives with probability $\overline{\delta}_i$ after each period, a fraction $(1-\overline{\delta}_i)\overline{\delta}_i^{k-1}$ of active player i has age k. Using (3), the law of total probabilities and the fact that $\pi_k \in [0, 1]$ for every

$k \in \mathbb{N}$, we have

$$1-\Pi_i^{(\sigma,\mu)}(a_i^*)=\sum_{k=1}^{+\infty}(1-\overline{\delta}_i)\overline{\delta}_i^{k-1}\pi_k \leq 1-\overline{\delta}_i^{\sum_{k=1}^{+\infty}\pi_k} \leq 1-\overline{\delta}_i^{\frac{\overline{u}_i-\underline{u}_i}{c^*}\cdot\frac{\delta_i}{1-\delta_i}}. \tag{4}$$

Since $\delta_i \equiv \overline{\delta}_i \cdot \widehat{\delta}_i$, once we fix any $\widehat{\delta}_i \in (0,1)$ and let $\overline{\delta}_i \to 1$, the RHS of (4) will converge to 0.

Second, suppose player i can add signals and f_i has non-shifting support. For every $r_i \in R_i^*$ and $s_i \in S_i$, let $\widehat{R}_i(r_i)$ denote the set of $r_i' \in R_i^*$ such that r_i is a subsequence of r_i', and let $\widehat{R}_i(r_i,s_i)$ denote the set of $r_i' \in R_i^*$ such that (r_i,s_i) is a subsequence of r_i'. Fix $r_i \in R_i^*$, player i's continuation value in the next period is $\max_{r_i' \in \widehat{R}_i(r_i,s_i)} V(r_i')$ when (i) his current-period record is r_i and (ii) his action and his current-period partner's generate signal s_i.[8] Since $\widehat{R}_i(r_i,s_i) \subset \widehat{R}_i(r_i)$, it must be the case that $V(r_i) \geq \max_{r_i' \in \widehat{R}_i(r_i,s_i)} V(r_i')$. This is because otherwise, player i will have a strict incentive to deviate by choosing $\arg\max_{r_i' \in \widehat{R}_i(r_i,s_i)} V(r_i')$ before being matched with his current-period partner instead of choosing r_i, which violates the hypothesis that $r_i \in R_i^*$. This suggests that player i's continuation value is *non-increasing* over time. When player i with record r_i is matched with record r_{-i}, he has an incentive to play $a_i' \neq a_i^*$ only if

$$\begin{aligned}&(1-\delta_i)u_i(a_i',\sigma_{-i}^a(r_i,r_{-i}))+\delta_i\sum_{s_i\in S_i} f_i(s_i|a_i',\sigma_{-i}^a(r_i,r_{-i}))\max_{r_i'\in\widehat{R}_i(r_i,s_i)} V(r_i')\\ &\quad\geq (1-\delta_i)u_i(a_i^*,\sigma_{-i}^a(r_i,r_{-i}))+\delta_i\sum_{s_i\in S_i} f_i(s_i|a_i^*,\sigma_{-i}^a(r_i,r_{-i}))\max_{r_i'\in\widehat{R}_i(r_i,s_i)} V(r_i').\end{aligned}\tag{5}$$

Inequality (5) together with the fact that $V(r_i) \geq \max_{r_i' \in \widehat{R}_i(r_i,s_i)} V(r_i')$ for every $s_i \in S_i$ implies that

$$\begin{aligned}&(1-\delta_i)u_i(a_i',\sigma_{-i}^a(r_i,r_{-i}))+\delta_i V(r_i)\\ &\quad\geq (1-\delta_i)u_i(a_i^*,\sigma_{-i}^a(r_i,r_{-i}))+\delta_i\sum_{s_i\in S_i} f_i(s_i|a_i^*,\sigma_{-i}^a(r_i,r_{-i}))\max_{r_i'\in\widehat{R}_i(r_i,s_i)} V(r_i').\end{aligned}$$

The above inequality together with the definition of c^* in (1) implies that

$$\sum_{s_i\in S_i} f_i(s_i|a_i^*,\sigma_{-i}^a(r_i,r_{-i}))\left\{V(r_i)-\max_{r_i'\in\widehat{R}_i(r_i,s_i)} V(r_i')\right\} \geq \frac{1-\delta_i}{\delta_i}c^*. \tag{6}$$

Since f_i has non-shifting support, the set $S_i(a_{-i}) \equiv \{s_i \in S_i | f_i(s_i|a_i,a_{-i})>0\}$ is well defined, so

$$q(f_i) \equiv \min_{a_i'\neq a_i^*, a_{-i}\in A_{-i}, s_i\in S_i(a_{-i})} \frac{f_i(s_i|a_i',a_{-i})}{f_i(s_i|a_i^*,a_{-i})}$$

8. The hypothesis that (σ,μ) is an equilibrium implies that player i has at least one best reply at every positive probability information set where he needs to choose which additional signals to include in his record. Hence, $\max_{r_i' \in \widehat{R}_i(r_i,s_i)} V(r_i')$ exists.

is strictly positive. Since $V(r_i) - \max_{r_i' \in \widehat{R}_i(r_i,s_i)} V(r_i') \geq 0$ for every $s_i \in S_i$, inequality (6) implies that

$$\sum_{s_i \in S_i} f_i(s_i|a_i', \sigma_{-i}^a(r_i, r_{-i})) \left\{ V(r_i) - \max_{r_i' \in \widehat{R}_i(r_i,s_i)} V(r_i') \right\} \geq \frac{1-\delta_i}{\delta_i} c^* q(f_i) \quad \text{for every } a_i' \in A_i. \tag{7}$$

Since (7) holds at every (r_i, r_{-i}) where player i plays $a_i' \neq a_i^*$ with positive probability, the definitions of V_k, V_{k+1}, and π_k imply that $V_k - V_{k+1} \geq \frac{1-\delta_i}{\delta_i} c^* q(f_i) \pi_k$. Summing this inequality up for all $k \in \mathbb{N}$, we have $\frac{1-\delta_i}{\delta_i} c^* q(f_i) \sum_{k=1}^{+\infty} \pi_k \leq \sum_{k=1}^{+\infty} (V_k - V_{k+1}) \leq \overline{V} - \underline{V} \leq \overline{u}_i - \underline{u}_i$. Similar to the derivation of (4),

$$1 - \Pi_i^{(\sigma,\mu)}(a_i^*) = \sum_{k=1}^{+\infty} (1 - \overline{\delta}_i) \overline{\delta}_i^{k-1} \pi_k \leq 1 - \overline{\delta}_i^{\sum_{k=1}^{+\infty} \pi_k} \leq 1 - \overline{\delta}_i^{\frac{\overline{u}_i - \underline{u}_i}{c^* q(f_i)} \cdot \frac{\delta_i}{1-\delta_i}}. \tag{8}$$

The RHS of (8) vanishes to 0 once we fix any $\widehat{\delta}_i \in (0, 1)$ and f_i with non-shifting support and let $\overline{\delta}_i \to 1$.

4. COOPERATION BETWEEN PLAYERS WITH INTERMEDIATE EXPECTED LIFESPANS

Theorem 1 implies that sufficiently long-lived players will almost never cooperate. Sufficiently short-lived players have no incentive to cooperate since their effective discount factors are too low. The rest of this section examines whether players with *intermediate* expected lifespans can sustain cooperation.

I start from showing that the average probability of cooperation is uniformly bounded below 1 for *all* $\overline{\delta}_i$ and $\widehat{\delta}_i$. Inequalities (4) and (8) imply that player i's average probability of taking actions other than his dominant action a_i^* is no more than $1 - \overline{\delta}_i^{\frac{\overline{u}_i - \underline{u}_i}{x} \cdot \frac{\delta_i}{1-\delta_i}}$, where (i) $x = c^*$ when player i can erase signals and (ii) $x = c^* q(f_i)$ when player i can add signals and f_i has non-shifting support. Since $\frac{\delta_i \log \delta_i}{1-\delta_i}$ is decreasing in $\delta_i \in [0, 1)$,

$$\frac{\delta_i \log \overline{\delta}_i}{1 - \delta_i} \geq \frac{\delta_i \log \delta_i}{1 - \delta_i} \geq \lim_{\delta \to 1} \frac{\delta \log \delta}{1 - \delta} = -1 \quad \text{for every } \delta_i \in [0, 1). \tag{9}$$

Therefore, fixing the stage-game payoff u_i and the signal distribution f_i, we know that for all $\overline{\delta}_i$ and $\widehat{\delta}_i$,

$$1 - \Pi_i^{(\sigma,\mu)}(a_i^*) \leq 1 - \exp\left(-\frac{\overline{u}_i - \underline{u}_i}{x}\right) \quad \text{in every equilibrium } (\sigma, \mu). \tag{10}$$

Inequality (10) implies that it is impossible to sustain full cooperation in any equilibrium either when players can erase signals or when players can add signals and their signal distribution has non-shifting support. The rest of this section examines whether *some* cooperation can be sustained in *some* equilibria.

I focus on equilibria that are *purifiable* as in Bhaskar and Thomas (2019) since mixed-strategy equilibria may not be robust to private payoff information. Formally, I refer to $\mathcal{G} = (I, A, u)$ as the *unperturbed stage game*. For every $\varepsilon > 0$, an *ε-perturbed stage game* $\mathcal{G}(\varepsilon) \equiv (I, A, u^\varepsilon)$ is one where player i's period k stage-game payoff is $u_i^\varepsilon(a_i, a_{-i}) \equiv$

$u_i(a_i, a_{-i}) + \varepsilon z_{i,k}(a_i)$, where $z_{i,k}(a_i)$ is a random payoff shock. The shocks $z_{i,k}(a_i)$ are i.i.d. across actions, players, and periods. Before player i acts in period k, he observes the realizations of his own current-period payoff shocks $\{z_{i,k}(a_i)\}_{a_i \in A_i}$ but *not* the ones in the future and of others. An equilibrium (σ, μ) of the unperturbed repeated game is *purifiable* if fixing the distribution of $z_{i,k}(a_i)$ that has *bounded support* and *no atom*, for every sequence $\varepsilon_n \to 0$, there exist a sequence of equilibria $(\sigma(\varepsilon_n), \mu(\varepsilon_n))_{n \in \mathbb{N}}$ of the repeated ε_n-perturbed stage games $\mathcal{G}(\varepsilon_n)$ that converge to (σ, μ).

As in the main results of Takahashi (2010), Heller and Mohlin (2018), and Clark *et al.* (2021), I restrict attention to records that are *first-order*, namely, the distribution of s_i, denoted by $f_i(\cdot|a_i, a_{-i})$, is independent of a_{-i}.[9] I will write $f_i(\cdot|a_i)$ instead of $f_i(\cdot|a_i, a_{-i})$ in order to avoid cumbersome notation. I will state my results in the prisoner's dilemma,[10] with stage-game payoffs given by

	Cooperate	Defect
Cooperate	$1, 1$	$-l, 1+g$
Defect	$1+g, -l$	$0, 0$

with $g, l > 0$.

In the above prisoner's dilemma, I say that players' payoffs are *weakly supermodular* if $g \leq l$, are *strictly supermodular* if $g < l$, are *weakly submodular* if $g \geq l$, and are *strictly submodular* if $g > l$.

Theorem 2 shows that when players have *intermediate* expected lifespans and their actions are *precisely monitored*, some cooperation can be sustained in purifiable equilibria *either* when players can *only* erase signals and their payoffs are strictly submodular, *or* when they can *only* add signals and their payoffs are strictly supermodular. For every $\varepsilon > 0$, I say that the monitoring structure is ε-precise if for every $i \in \{1, 2\}$, there exists $s_i^* \in S_i$ such that $f_i(s_i^*|a_i = C) \geq 1 - \varepsilon$ and $f_i(s_i^*|a_i = D) \leq \varepsilon$. Hence, perfect monitoring is ε-precise. For each $\varepsilon > 0$, there exists f_i that is first order, ε-precise, and satisfies non-shifting support.

Theorem 2. *Suppose all players' records are first-order, either they can only erase signals and have strictly submodular payoffs, or they can only add signals and have strictly supermodular payoffs. There exist $\delta^* \in (0, 1)$ and $\varepsilon > 0$ such that when the monitoring structure is ε-precise and $\widehat{\delta}_1, \widehat{\delta}_2 > \delta^*$, there exists a non-empty interval $[\delta', \delta''] \subset (0, 1)$ such that as long as $\overline{\delta}_1, \overline{\delta}_2 \in [\delta', \delta'']$, there exists a purifiable equilibrium (σ, μ) such that $\Pi_i^{(\sigma,\mu)}(C) > 0$ for every $i \in \{1, 2\}$.*

I explain the ideas behind the proof with details relegated to Appendix B. When players can only erase signals and have strictly submodular payoffs, I categorize them into *juniors* who have no s_i^* in their records and *seniors* who have at least one s_i^*. Seniors always defect. Juniors cooperate with seniors for sure and cooperate with other juniors with probability $q_j \in (0, 1)$. Since payoffs are strictly submodular, if juniors are indifferent between C and D when facing other juniors, then they strictly prefer C when facing seniors.

When players can only add signals and their payoffs are strictly supermodular, I categorize them into *juniors* with no bad signal (*i.e.* signals other than s_i^*) in their records and *seniors* with at least one bad signal in their records. In equilibrium, seniors defect against everyone, juniors

9. The "first-order record" requirement can be relaxed for Theorem 2, but it will be needed for Theorem 3.

10. In a working paper version (Pei, 2024), I focus on general monotone games when players' expected lifespans are intermediate.

cooperate with other juniors with probability $q_i \in (0, 1)$, and defect for sure against seniors. Since payoffs are strictly supermodular, if juniors are indifferent between C and D when facing juniors, then they strictly prefer D when facing seniors.

These strategy profiles are part of some purifiable equilibria when $\overline{\delta}_1$ and $\overline{\delta}_2$ are intermediate but *not* when they are close to 1. This is because when players are sufficiently long-lived, there will be too few juniors in the population to provide incentives, in which case juniors will have no incentive to cooperate.

Theorem 3 shows that the conditions in Theorem 2 on players' payoffs are essential in the sense that when players can erase signals and have weakly supermodular payoffs, or can add signals and have weakly submodular payoffs, all players will always defect regardless of $(\overline{\delta}_1, \overline{\delta}_2, \widehat{\delta}_1, \widehat{\delta}_2)$ and the signal precision.

Theorem 3. *If players' records are first order and either (i) payoffs are weakly supermodular and one population i can erase signals or (ii) payoffs are weakly submodular, and one population i can add signals and f_i has non-shifting support,*[11] *then* $\Pi_1^{(\sigma,\mu)}(C) = \Pi_2^{(\sigma,\mu)}(C) = 0$ *for every purifiable equilibrium* (σ, μ).

The proof is in Appendix C, which uses the purifiability refinement as well as the fact that the matching is uniform. To see why, let us start from the case where players in population 2 can erase signals. Pick any player 1 and compare his incentives to cooperate with (i) player 2 who has the highest continuation value (call him player 2^*) and (ii) any other player 2. Player 2^* will defect for sure due to his ability to erase signals. If players' actions are complements (*i.e.* $g \leq l$), any player 1 will have less incentive to cooperate with player 2^* than with any other player 2. If this is the case, then it is impossible to deliver player 2^* a strictly higher continuation value than to any other player 2.[12] This will break down cooperation.

Similarly, when players in population 2 can add signals, player 2 with the lowest continuation value (call him player $2^\dagger$) will defect for sure. If players' actions are substitutes (*i.e.* $g \geq l$), then any player 1 will have weakly stronger incentives to cooperate with player $2^\dagger$ than with any other player 2. This again implies that in all purifiable equilibria, it is impossible to deliver strictly lower continuation values to player $2^\dagger$ than to any other player 2. As a result, players will have no incentive to cooperate due to the lack of punishments.[13]

5. EXTENSIONS

Theorem 1 extends to other settings where players' continuation value is either non-decreasing or non-increasing over time. For example, player i's continuation value is non-increasing over time (i) when the additional signals he included in his current-period record do not have to appear in his future records, or (ii) before being matched with each new partner, all of a player's

11. Recall that the non-shifting support condition only requires that the *support* of player i's signal distribution to be independent of his actions. Hence, for every $\varepsilon > 0$, there exist signal distributions that are first order, ε-precise, and satisfy non-shifting support.

12. Purifiability is needed since in some non-purifiable equilibria, there exists player 1's record r_1 such that (i) all players from population 2 will defect against player 1 with record r_1 but (ii) r_1 will cooperate with player 2 who has the highest continuation value and will sometimes defect against player 2 with lower continuation values. This cannot happen in purifiable equilibria as players' behaviours can condition *only* on payoff-relevant information. See Bhaskar (1998) and Bhaskar *et al*. (2013).

13. The non-shifting support condition is required when players can only add signals, since my argument only implies that player 2's highest and lowest continuation values *at positive probability histories* must coincide in all purifiable equilibria. Under perfect monitoring, it cannot rule out grim-trigger equilibria since there will be only one continuation value on the equilibrium path.

newly added signals in the current period must come *after* all the signals he generated and added before (in the sequence of signals).

In a working paper version (Pei, 2024), I present a model with general record systems where players' continuation value is non-decreasing over time. This includes, for example, when players can only erase their most recent signal but not the ones they generated before, and when they can choose a subset of the signals generated by their past actions and disclose those signals to their current partner, *i.e.* the signals they did not disclose to their period-k partner can be disclosed to their period-$(k+1)$ partner.

Theorem 1 extends to settings where players can only observe a garbled version of their partners' records, such as when players *cannot* observe the exact sequence of signals in their partners' records and can only observe some summary statistics (*e.g.* the number of times that each signal realization occurred in the player's record). It also extends to settings where players cannot perfectly control their partners' observations, such as each player observes his partners' records with some idiosyncratic observational noise.

Next, I extend Theorem 1 to settings where players' continuation value is *not necessarily monotone*. This includes, for example, when players choose their actions, they face uncertainty regarding whether they can erase or add signals before being matched with their next partner. If this is the case, then players do not know whether they can preserve their current record in the next period and whether their next-period continuation value is no less than that in the period after next. The extension below nests my baseline model.

Formally, suppose whether each player i can erase or add signals before being matched with his period-k partner is determined by an i.i.d. random variable $m_{i,k} \in \{\text{erase}, \text{add}, \text{both}, \text{none}\}$,[14] where $p_e \equiv \Pr(m_{i,k} = \text{erase})$, $p_a \equiv \Pr(m_{i,k} = \text{add})$, $p_b \equiv \Pr(m_{i,k} = \text{both})$, and $p_n \equiv \Pr(m_{i,k} = \text{none})$. Player i observes the realization of $m_{i,k}$ when he chooses his period-k record *before* being matched with his period-k partner.

For any player i born before period k, suppose his period-$(k-1)$ record is $r_{i,k-1}$ and his period-$(k-1)$ signal is $s_{i,k-1}$. If $m_{i,k} = \text{erase}$, then he can choose his period-k record from any subsequence of $(r_{i,k-1}, s_{i,k-1})$. If $m_{i,k} = \text{add}$, then he can choose his period-k record to be any r_i' such that $(r_{i,k-1}, s_{i,k-1})$ is a subsequence of r_i'. If $m_{i,k} = \text{both}$, then he can choose his period-k record from the entire set R_i. If $m_{i,k} = \text{none}$, then his record in period k must be $(r_{i,k-1}, s_{i,k-1})$. For any player i born in period k, before being matched with his partner in period k, he can choose anything in R_i to be his period-k record if $m_{i,k} \in \{\text{add}, \text{both}\}$ and can only choose $\varnothing$ to be his period-k record if $m_{i,k} \in \{\text{erase}, \text{none}\}$.

Theorem 4. *Suppose players in population i have a strictly dominant action $a_i^* \in A_i$.*

(1) *For every $\varepsilon > 0$, there exists $p^* \in (0,1)$ such that when $p_b + p_e \geq p^*$, for every $\widehat{\delta}_i \in (0,1)$, there exists $\delta^* \in (0,1)$ such that $\Pi_i^{(\sigma,\mu)}(a_i^*) \geq 1-\varepsilon$ for every equilibrium (σ,μ) when $\overline{\delta}_i > \delta^*$.*

(2) *For every $\varepsilon > 0$, there exists $p^* \in (0,1)$ such that when $p_b + p_a \geq p^*$, for every $\widehat{\delta}_i \in (0,1)$ and f_i that has non-shifting support, there exists $\delta^* \in (0,1)$ such that $\Pi_i^{(\sigma,\mu)}(a_i^*) \geq 1-\varepsilon$ for every equilibrium (σ,μ) when $\overline{\delta}_i > \delta^*$.*

14. My result in this section, Theorem 4, applies both to the case where all players in population i face the same shock (*i.e.* in any given period, either all of them can only erase signals, or all of them can only add signals, or all of them can do both, or all of them cannot manipulate) and to the case where the shocks are i.i.d. across periods and players within population i.

Theorem 4 implies that if player i has a dominant action and can either erase or add signals *with probability above some cutoff*, then his average probability of cooperation is low when he is sufficiently long-lived.

Appendix D shows the first statement of Theorem 4, that is, the case where player i can erase signals with probability above p^*. The case where player i can add signals with high probability and f_i having non-shifting support can be shown using a similar argument, which I omit in order to avoid repetition.

The main challenge is that unlike in the baseline model, it is unclear whether player i's continuation value will increase or decrease over time: it may increase in periods where $m_{i,k} \in \{\text{erase, none}\}$ and it may decrease in periods where $m_{i,k} \in \{\text{add, none}\}$. As a result, players' continuation values are no longer monotone over time, which contrasts to the baseline model. In addition, players may have incentives to cooperate both when their continuation values reach the maximum and when their continuation values reach the minimum. Hence, their incentives to cooperate do not vanish over time, unlike in the baseline model.

To conclude this section, I consider situations where a third party (*e.g.* an online platform) can *reset* a player's record. Such resets can be implemented either by *erasing* all signals from a player's record, or by *adding* some signal to a player's record (*e.g.* adding a signal that marks the beginning of a reset).

First, suppose player i can only erase signals and that after each period, a platform randomly selects a fraction $p_i \in [0, 1]$ of the active players in population i and erase all signals from their records.[15] Since a player's continuation value depends only on his record, once a player is selected by the platform, he will effectively become a newly born player in the next period, so each player's expected lifespan is shortened from $(1-\overline{\delta}_i)^{-1}$ to $(1-\overline{\delta}_i(1-p_i))^{-1}$. Although the extreme form of anti-folk theorem that players cooperate with probability close to 0 may not extend, my uniform upper bound (10) still applies: It implies that the average probability of cooperation is uniformly bounded below 1 regardless of p_i. Hence, (10) illustrates the limits of fostering cooperation via such policies and implies that the folk theorem fails more generally.

Second, suppose after each period, a platform randomly selects a fraction $p_i \in [0, 1]$ of the active players in population i and adds a signal s_i^* to their records. If player i can erase signals, then Theorem 1 extends since before being matched with each new partner, player i can always erase the signal added by the platform and preserve his record in the period before, so his continuation value is non-decreasing over time. If player i cannot erase signals but can add signals, then Theorem 1 extends since player i can replicate his future records in the current period (by adding s_i^* himself) so his continuation value is non-increasing over time.

6. CONCLUDING REMARKS

This paper establishes an anti-folk theorem in community enforcement models, which shows that *sufficiently long-lived players* will almost never cooperate. This result is driven by the *monotonicity constraints* on players' continuation values and these constraints are implied by players' abilities to add or erase signals.

In summary, when players' continuation values are *non-decreasing* over time (which must be the case when they can erase signals from their records), they will have no incentive to cooperate once their continuation values are close to the maximum. As a result, each player will

15. If player i can add signals, then he can always add the erased signals back, in which case Theorem 1 still applies.

only cooperate in a bounded number of periods and these periods carry negligible weight once the player is sufficiently long-lived.

When players' continuation values are *non-increasing* over time (which must be the case when they can add signals), they will have no incentive to cooperate once their continuation values approach the minimum. If the signals that monitor players' actions are noisy, then in order to provide incentives to cooperate, players' continuation value needs to decrease significantly relative to their current continuation value regardless of the actions they take. This again suggests that each player has an incentive to cooperate for at most a bounded number of periods and these periods carry negligible weight once the player is sufficiently long-lived.

My approach to endogenous records has two limitations. First, Theorems 1 and 4 require $\overline{\delta}_i$ to be close to 1. When $\overline{\delta}_i$ is bounded below 1, although I can derive a uniform upper bound on the average probability of cooperation, there is no precise characterization of the maximal probability of cooperation. Computing such a probability is hard even for the prisoner's dilemma since very little is known about (i) the set of equilibria in repeated games when δ_i is bounded below 1 and (ii) the structure of *steady-state* Nash equilibria.

Second, when player i can only add signals, I focus on the case where he can include *any* finite number of signals in any period. This modelling assumption ensures that his continuation value is non-increasing over time. However, if player i can add no more than K signals in each period, his continuation value may not be non-increasing over time. Whether Theorem 1 holds for a bounded K remains an open question.

APPENDIX

A. Proof of Corollary 1

Recall that in any *finite* 2-player game $\mathcal{G}$, a pure action $a_i \in A_i$ is strictly dominated in the stage game *if and only if* it is never a best reply and there exists $\eta > 0$ such that regardless of player $-i$'s action $\alpha_{-i} \in \Delta(A_{-i})$, player i's payoff from playing a_i is less than his payoff from playing a best reply minus η.

Using the same argument as that in the proof of Theorem 1, one can show that for every $\widehat{\delta}_i \in (0, 1)$, there exists $\delta^* \in (0, 1)$ such that when $\overline{\delta}_i > \delta^*$, the average probability that player i takes strictly dominated actions is less than ε in all equilibria. Let $A_i^1 \subset A_i$ denote the set of player i's actions that survive the first round of deletion but not the second round. If A_i^1 is non-empty, then there exists $\eta > 0$ that depends only on u_i such that all actions in A_i^1 are still strictly dominated by at least η when the probability that player $-i$ takes strictly dominated actions is no more than η. According to the Markov's inequality, if the average probability with which player $-i$ takes strictly dominated actions is no more than ε, then histories where player $-i$ takes strictly dominated actions with probability more than η occurs with probability less than ε/η. Using the argument in Theorem 1, we know that for every $\widehat{\delta}_i \in (0, 1)$, there exists $\delta^* \in (0, 1)$ such that when $\overline{\delta}_i > \delta^*$, the probability that player i takes actions in A_i^1 is at most $\varepsilon + \varepsilon/\eta$. The conclusion of Corollary 1 is obtained once we iterate the above process for at most $|A_1| + |A_2|$ rounds.

B. Proof of Theorem 2

I focus on the case where players can erase signals and have strictly submodular payoffs. The case where players can add signals and have strictly supermodular payoffs is symmetric, which I omit to avoid repetition.

To simplify notation, let $f_i^* \equiv f_i(s_i^*|a_i = C)$ and $f_i' \equiv f_i(s_i^*|a_i = D)$. There exists $\overline{\varepsilon} > 0$ such that for every $\varepsilon < \overline{\varepsilon}$, $f_i^* > f_i'$ when (f_1, f_2) is ε-precise. Let $\overline{V}_i$ denote the continuation value of seniors in population i and let $\underline{V}_i$ denote the continuation value of juniors in population i. Let q_i denote the probability that juniors play C against other juniors and let μ_i denote the fraction of players in population i that are juniors. A junior in population i's indifference condition when facing a junior in population j is given by

$$(1 - \delta_i)u_i(C, q_j C + (1 - q_j)D) + \delta_i(f_i^* \overline{V}_i + (1 - f_i^*)\underline{V}_i) = (1 - \delta_i)u_i(D, q_j C + (1 - q_j)D)$$
$$+ \delta_i(f_i' \overline{V}_i + (1 - f_i')\underline{V}_i),$$

which implies that

$$\overline{V}_i - \underline{V}_i = \frac{1-\delta_i}{\delta_i} \cdot \frac{1}{f_i^* - f_i'} \cdot \left(q_j g + (1-q_j)l\right). \tag{B.1}$$

Since it is always weakly optimal for juniors to cooperate, his continuation value $\underline{V}_i$ can be written as

$$\underline{V}_i = \mu_j \left\{(1-\delta_i)u_i(C, q_j C + (1-q_j)D) + \delta_i \left(f_i^* \overline{V}_i + (1-f_i^*)\underline{V}_i\right)\right\} + (1-\mu_j)\left\{(1-\delta_i)u_i(C,D) + \delta_i \left(f_i^* \overline{V}_i + (1-f_i^*)\underline{V}_i\right)\right\}.$$

Plugging in (B.1) for the difference between $\overline{V}_i$ and $\underline{V}_i$, we obtain that

$$\underline{V}_i = \mu_j q_j - (1-\mu_j q_j)l + \frac{f_i^*}{f_i^* - f_i'}\Big(q_j g + (1-q_j)l\Big). \tag{B.2}$$

A senior's continuation value is $\overline{V}_i = \mu_j(1+g)$. This together with equations (B.1) and (B.2) implies that

$$\mu_j(1+g) = \mu_j q_j - (1-\mu_j q_j)l + \frac{f_i^*}{f_i^* - f_i'}\Big(q_j g + (1-q_j)l\Big) + \frac{1-\delta_i}{\delta_i} \cdot \frac{1}{f_i^* - f_i'} \cdot \Big(q_j g + (1-q_j)l\Big). \tag{B.3}$$

For every $\overline{\eta} > 0$, there exists $\varepsilon > 0$ such that as long as monitoring is ε-precise, there exists $\eta \in (-\overline{\eta}, \overline{\eta})$ such that

$$(1-q_j)\mu_j = \frac{1-\delta_i}{\delta_i} \cdot \frac{l}{1+g} + q_j \frac{g-l}{1+g}\Big(\delta_i^{-1} - \mu_j\Big) + \eta. \tag{B.4}$$

Since $g > l$, the LHS of (B.4) is strictly decreasing in q_j and equals 0 when $q_j = 1$ and the RHS of (B.4) is strictly increasing in q_j and is always strictly positive when $|\eta| < \frac{l}{1+g} \cdot \frac{1-\delta_i}{\delta_i}$. This implies that there exists a solution to (B.4) for η small enough if and only if the LHS is greater than the RHS when $q_j = 0$, or equivalently,

$$\mu_j > \frac{1-\delta_i}{\delta_i} \cdot \frac{l}{1+g} \text{ for every } i, j \in \{1,2\} \text{ with } i \neq j. \tag{B.5}$$

where steady-state record distributions, μ_i and μ_j, must satisfy

$$\mu_i = (1-\overline{\delta}_i) + \overline{\delta}_i \Big\{\mu_i \mu_j \big((1-q_i)(1-f_i') + q_i(1-f_i^*)\big) + \mu_i(1-\mu_j)(1-f_i^*)\Big\}. \tag{B.6}$$

Equation (B.6) implies that $\mu_i \geq 1 - \overline{\delta}_i$ for every $i \in \{1,2\}$, and therefore, it is sufficient to show that there exists a non-empty interval $[\delta', \delta''] \subset (0,1)$ such that when $\overline{\delta}_1, \overline{\delta}_2 \in [\delta', \delta'']$, we have

$$\mu_j \geq 1 - \overline{\delta}_j > \frac{1-\delta_i}{\delta_i} \cdot \frac{l}{1+g}. \tag{B.7}$$

This is indeed the case when $\widehat{\delta}_i$ is close enough to 1 for every $i \in \{1,2\}$, under which (B.7) is satisfied as long as $\overline{\delta}_j$ is not too close to 1. Such an equilibrium is purifiable since players have strict incentives at all except for one information set, which is when a junior is matched with another junior.

C. Proof of Theorem 3

Fix any equilibrium (σ, μ). Let $R_i^* \subset R_i$ denote the set of player i's records that occur with positive probability under μ. Let $V_i(r_i)$ denote player i's continuation value when his record is r_i before knowing his current partner's record. Let $\overline{V}_i \equiv \sup_{r_i \in R_i^*} V_i(r_i)$ and let $\underline{V}_i \equiv \inf_{r_i \in R_i^*} V_i(r_i)$, which are player i's highest and lowest continuation values, respectively, at records that occur with positive probability under μ.

Lemma 1. *Suppose players in population 1 can erase signals or they can add signals and f_1 has non-shifting support. If there exist an equilibrium (σ, μ) and a record profile $(r_1, r_2) \in R_1^* \times R_2^*$ such that player 1 plays C with positive probability at (r_1, r_2), then it must be the case that $\overline{V}_1 > \underline{V}_1$.*

Proof: First, suppose players in population 1 can erase signals. For each player 1, his incentive to play C at $(r_1, r_2) \in R_1^* \times R_2^*$ implies that

$$(1-\delta_1)u_1(C, \sigma_2^a(r_1, r_2)) + \delta_1 \overline{V}_1 \geq (1-\delta_1)u_1(D, \sigma_2^a(r_1, r_2)) + \delta_1 V_1(r_1). \tag{C.1}$$

Therefore, $\overline{V}_1 > V_1(r_1) \geq \underline{V}_1$ where the last inequality comes from the hypothesis that $r_1 \in R_1^*$.

Next, suppose players in population 1 can add signals. For each player 1, his incentive to play C at $(r_1, r_2) \in R_1^* \times R_2^*$ implies that

$$\begin{aligned}(1-\delta_1)u_1(C, \sigma_2^a(r_1, r_2)) &+ \delta_1 \sum_{s_1 \in S_1} f_1(s_1|C, \sigma_2^a(r_1, r_2)) \max_{r_1' \in \widehat{R}_1(r_1, s_1)} V_1(r_1') \\ &\geq (1-\delta_1)u_1(D, \sigma_2^a(r_1, r_2)) + \delta_1 \sum_{s_1 \in S_1} f_1(s_1|D, \sigma_2^a(r_1, r_2)) \max_{r_1' \in \widehat{R}_1(r_1, s_1)} V_1(r_1'). \end{aligned} \tag{C.2}$$

If f_1 has non-shifting support, let $S_1(\sigma_2^a(r_1, r_2))$ denote the set of player 1's signals that occur with positive probability when player 2's action is $\sigma_2^a(r_1, r_2)$. This set is well defined in the sense that it does not depend on player 1's action. Inequality (C.2) and the definitions of $\overline{V}_1, \underline{V}_1$ suggest that

$$\overline{V}_1 \geq \max_{s_1 \in S_1(\sigma_2^a(r_1, r_2))} \left\{ \max_{r_1' \in \widehat{R}_1(r_1, s_1)} V_1(r_1') \right\} > \min_{s_1 \in S_1(\sigma_2^a(r_1, r_2))} \left\{ \max_{r_1' \in \widehat{R}_1(r_1, s_1)} V_1(r_1') \right\} \geq \underline{V}_1.$$

Suppose by way of contradiction that under the conditions of Theorem 3, there exists a purifiable equilibrium (σ, μ) such that $\overline{V}_1 > \underline{V}_1$. Lemma 1 implies that any contradiction derived from this hypothesis will imply Theorem 3. Let

$$\eta \equiv \min \left\{ \frac{\overline{V}_1 - \underline{V}_1}{3}, \frac{1-\delta_1}{2\delta_1} c^* \right\}, \tag{C.3}$$

which is strictly positive since $\overline{V}_1 > \underline{V}_1$. The definitions of $\overline{V}_1$ and $\underline{V}_1$ imply that there exist $\overline{r}_1, \underline{r}_1 \in R_1^*$ with $V_1(\overline{r}_1) \in [\overline{V}_1 - \eta, \overline{V}_1]$ and $V_1(\underline{r}_1) \in [\underline{V}_1, \underline{V}_1 + \eta]$. The definition of η implies that $V_1(\overline{r}_1) > V_1(\underline{r}_1)$. Recall from Section 3.1 that when player 1 can erase signals, his equilibrium continuation value is non-decreasing over time, and that when player 1 can add signals, his equilibrium continuation value is non-increasing over time. Lemma 2 is then implied by (C.1) and (C.2) as well as the monotonicity constraints.

Lemma 2. *If population 1 can erase signals, then they have strict incentives to play D at any record* $\overline{r}_1 \in R_1^*$ *that satisfies* $V_1(\overline{r}_1) \in [\overline{V}_1 - \eta, \overline{V}_1]$. *If population 1 can add signals and* f_1 *has non-shifting support, then they have strict incentives to play D at any record* $\underline{r}_1 \in R_1^*$ *that satisfies* $V_1(\underline{r}_1) \in [\underline{V}_1, \underline{V}_1 + \eta]$.

Lemma 3 establishes an implication of the purifiability refinement.

Lemma 3. *Suppose players in population 2 have the first-order records.*

If players in population 1 can erase signals and $g \leq l$, *then in every purifiable equilibrium, for every* $\overline{r}_1 \in R_1^*$ *that satisfies* $V_1(\overline{r}_1) \geq \overline{V}_1 - \eta$, *every* $r_1 \in R_1^*$, *and every* $r_2 \in R_2^*$, *each player 2's probability of playing C at* $(\overline{r}_1, r_2)$ *is weakly less than his probability of playing C at* (r_1, r_2).

If players in population 1 can add signals and $g \geq l$, *then in every purifiable equilibrium, for every* $\underline{r}_1 \in R_1^*$ *that satisfies* $V_1(\underline{r}_1) < \underline{V}_1 + \eta$, *every* $r_1 \in R_1^*$, *and every* $r_2 \in R_2^*$, *each player 2's probability of playing C at* $(\underline{r}_1, r_2)$ *is weakly more than his probability of playing C at* (r_1, r_2).

Proof: I will only prove the first statement. The proof of the second statement is symmetric, which I omit in order to avoid repetition. Lemma 2 implies that player 1 has a strict incentive to play D when his record is $\overline{r}_1$. Since player 2's records are first order, his payoff from playing a_2 at $(\overline{r}_1, r_2)$ is $(1-\delta_2)u_2(D, a_2) + \delta_2 \mathbb{E}[V_2|a_2, r_2]$, where $\mathbb{E}[V_2|a_2, r_2]$ stands for player 2's continuation value in the next period given his current-period action and record. Fix any $(r_1, r_2) \in R_1^* \times R_2^*$, if player 1's (possibly mixed) action at (r_1, r_2) is $\alpha_1 \in \Delta\{C, D\}$, then player 2's payoff from playing a_2 at (r_1, r_2) is $(1-\delta_2)u_2(\alpha_1, a_2) + \delta_2 \mathbb{E}[V_2|a_2, r_2]$. Since $g \leq l$, $\max_{a_2 \in A_2}\{(1-\delta_2)u_2(a_1, a_2) + \delta_2 \mathbb{E}[V_2|a_2, r_2]\}$ is a single-crossing function of a_1. Theorem 5 in Milgrom and Shannon (1994) implies that once we order players' actions by $C \succ D$, the set of maximizers when $a_1 = \alpha_1$, denoted by A_2^{**}, dominates the set of maximizers when $a_1 = D$, denoted by A_2^*, in strong set order. Consider any ε-perturbed stage game where player 2's stage-game payoff from playing a_2 is $u_2(a_1, a_2) + \varepsilon z_2(a_2)$, where $z_2(a_2)$ has bounded support and a non-atomic distribution. Since $z_2(a_2)$ has bounded support, player 2 will only take actions in A_2^* with positive probability at $(\overline{r}_1, r_2)$ and will only take actions in A_2^{**} with positive probability at (r_1, r_2) when ε is small enough. When $C \notin A_2^*$ or when $D \notin A_2^{**}$, the conclusion of Lemma 3 is trivially true. When $C \in A_2^*$ and $D \in A_2^{**}$, A_2^{**} dominates A_2^* in strong set order implies that $A_2^* = A_2^{**} = \{C, D\}$, in which case both the probability that player 2 plays C at $(\overline{r}_1, r_2)$ and the probability that player 2 plays C at (r_1, r_2) are between the probability of the event that $z_2(C) > z_2(D)$ and the probability of the event that $z_2(C) \geq z_2(D)$. Events $z_2(C) > z_2(D)$ and $z_2(C) \geq z_2(D)$ occur with the same probability when the distribution of z_2 is atomless. This implies the conclusion of Lemma 3.

I use Lemmas 1–3 to show Theorem 3. I consider two cases separately, depending on whether player 1 can erase or can add signals. This part of my proof uses the assumption that matching is uniform, and in particular, a player's record does not affect the distribution of opponents that he will be matched with.

Case 1

Suppose player 1 can erase signals. By Lemma 2, he will play D for sure at any record $\overline{r}_1$ that satisfies $V_1(\overline{r}_1) \geq \overline{V}_1 - \eta$. Therefore, his equilibrium continuation value at $\overline{r}_1$ equals

$$\sum_{r_2 \in R_2^*} \mu_2(r_2) \left\{(1-\delta_1) u_1(D, \sigma_2^a(\overline{r}_1, r_2)) + \delta_1 \mathbb{E}[V_1 | D, \overline{r}_1]\right\}. \tag{C.4}$$

For every $r_1 \in R_1^*$ with $V_1(r_1) < V_1(\overline{r}_1)$, if player 1's current-period record is r_1 and he deviates by playing D and erasing every signal he generates, then his record remains r_1, his payoff at r_1 under this deviation is

$$\sum_{r_2 \in R_2^*} \mu_2(r_2) \left\{(1-\delta_1) u_1(D, \sigma_2^a(r_1, r_2)) + \delta_1 V_1(r_1)\right\}, \tag{C.5}$$

which must be weakly lower than his equilibrium continuation value $V_1(r_1)$. According to Lemma 3, $\sigma_2^a(r_1, r_2)$ assigns weakly higher probability to C than $\sigma_2^a(\overline{r}_1, r_2)$, which implies that $u_1(a_1, \sigma_2^a(\overline{r}_1, r_2)) \leq u_1(a_1, \sigma_2^a(r_1, r_2))$ for every $a_1 \in \{C, D\}$. Since the difference between (C.4) and (C.5) is at least $V_1(\overline{r}_1) - V_1(r_1)$, one can obtain the following inequality by subtracting (C.5) from (C.4):

$$\mathbb{E}[V_1 | D, \overline{r}_1] - V_1(r_1) \geq \Big(V_1(\overline{r}_1) - V_1(r_1)\Big)\delta_1^{-1}. \tag{C.6}$$

Let $R_1(\overline{r}_1)$ denote the set of player 1's records that occur with positive probability in the next period when his current-period record is $\overline{r}_1$ and he plays his equilibrium strategy. Inequality (C.6) implies that

$$\max_{r_1' \in R_1(\overline{r}_1)} V_1(r_1') - V(r_1) \geq \mathbb{E}[V_1 | D, \overline{r}_1] - V_1(r_1) \geq \Big(V_1(\overline{r}_1) - V_1(r_1)\Big)\delta_1^{-1}. \tag{C.7}$$

Inequality (C.7) suggests that for any $\overline{r}_1 \in R_1^*$ that satisfies $V_1(\overline{r}_1) \in [\overline{V}_1 - \eta, \overline{V}_1]$, there exists $r_1' \in R_1(\overline{r}_1)$ such that $V_1(r_1') - V_1(r_1) \geq \delta_1^{-1}(V_1(\overline{r}_1) - V_1(r_1))$. This leads to a contradiction since there exists $\overline{r}_1 \in R_1^*$ that satisfies both $\overline{V}_1 > V_1(\overline{r}_1)$ and $\delta_1^{-1}(V_1(\overline{r}_1) - V_1(r_1)) > \overline{V}_1 - V_1(r_1)$ for any r_1 with $V_1(r_1) < V_1(\overline{r}_1)$, and any $r_1' \in R_1(\overline{r}_1)$ that satisfies $V_1(r_1') - V_1(r_1) \geq \delta_1^{-1}(V_1(\overline{r}_1) - V_1(r_1))$ will have $V_1(r_1') > \overline{V}_1$. Hence, $\Pi_1^{(\sigma,\mu)}(C) = 0$. The conclusion that $\Pi_1^{(\sigma,\mu)}(C) = 0$ then implies that $\Pi_2^{(\sigma,\mu)}(C) = 0$.

Case 2

Suppose player 1 can add signals and the distribution of f_1 has non-shifting support. For every record $\underline{r}_1 \in R_1^*$ that satisfies $V_1(\underline{r}_1) \leq \underline{V}_1 + \eta$, Lemma 2 implies that player 1 will play D for sure at $\underline{r}_1$, and Lemma 3 implies that for every $r_2 \in R_2^*$, the probability that player 2 plays C at $(\underline{r}_1, r_2)$ is weakly greater than the probability that he plays C at any (r_1, r_2). Since player 1's continuation value is non-increasing over time, starting from any such $\underline{r}_1$, any record of this player 1 that occurs with positive probability in the future, denote it by r_1', satisfies $V_1(r_1') \leq \underline{V}_1 + \eta$. Since once player 1's continuation value satisfies $V_1(r_1) \leq \underline{V}_1 + \eta$, his continuation value will also satisfy that in the future, we know that player 1's continuation value at $\underline{r}_1$ is weakly greater than his continuation value at every record in R_1^*. This contradicts the hypothesis that $\overline{V}_1 > \underline{V}_1$, and Lemma 1 then implies that $\Pi_1^{(\sigma,\mu)}(C) = 0$. The conclusion that $\Pi_1^{(\sigma,\mu)}(C) = 0$ then implies that $\Pi_2^{(\sigma,\mu)}(C) = 0$.

D. Proof of Theorem 4: Statement 1

Fix an equilibrium (σ, μ). Recall from the proof of Theorem 1 that R_i^* denotes the set of player i's records that occur with positive probability, $V(r_i)$ denotes player i's expected continuation value when his record is r_i, $R_i(r_i, s_i) \subset R_i^*$ denotes the set of subsequences of (r_i, s_i), and $\widehat{R}_i(r_i, s_i) \subset R_i^*$ denotes the set of r_i' such that (r_i, s_i) is a subsequence of r_i'. Let $\overline{u}_i$ and $\underline{u}_i$ denote player i's highest stage-game payoff and lowest stage-game payoff (as opposed to his minmax value), respectively. When $m_{i,k} =$ both, player i can choose any record from R_i before being matched with his partner in period k, so his continuation value in period k equals $\max_{r_i' \in R_i} V(r_i')$ regardless of whether he was born in period k as well as his record r_i and signal s_i in period $k-1$ (in the case where he was born before period k). The definition of c^* in (1) implies that for each player i, he has an incentive to play $a_i' \neq a_i^*$ at (r_i, r_{-i}) only if

$$
\begin{aligned}
& p_e \left\{ \sum_{s_i \in S_i} f_i(s_i | a_i', \sigma_{-i}^a(r_i, r_{-i})) \max_{r_i' \in R_i(r_i, s_i)} V(r_i') - \sum_{s_i \in S_i} f_i(s_i | a_i^*, \sigma_{-i}^a(r_i, r_{-i})) \max_{r_i' \in R_i(r_i, s_i)} V(r_i') \right\} \\
& + p_a \left\{ \sum_{s_i \in S_i} f_i(s_i | a_i', \sigma_{-i}^a(r_i, r_{-i})) \max_{r_i' \in \widehat{R}_i(r_i, s_i)} V(r_i') - \sum_{s_i \in S_i} f_i(s_i | a_i^*, \sigma_{-i}^a(r_i, r_{-i})) \max_{r_i' \in \widehat{R}_i(r_i, s_i)} V(r_i') \right\} \\
& + p_n \left\{ \sum_{s_i \in S_i} f_i(s_i | a_i', \sigma_{-i}^a(r_i, r_{-i})) V(r_i, s_i) - \sum_{s_i \in S_i} f_i(s_i | a_i^*, \sigma_{-i}^a(r_i, r_{-i})) V(r_i, s_i) \right\} \geq \frac{1-\delta_i}{\delta_i} c^*. \qquad \text{(D.1)}
\end{aligned}
$$

For every r_i and s_i, player i's continuation value at any record $r_i' \in \widehat{R}_i(r_i, s_i)$ is at least his payoff from the following strategy (i) not manipulating his record by the end of period k if $m_{i,k} \in \{\text{none, add}\}$, (ii) setting his record to r_i by erasing signals by the end of period k if $m_{i,k} \in \{\text{erase, both}\}$, and (iii) choosing a_i according to his equilibrium strategy at every record profile. This implies that

$$
V(r_i') \geq V(r_i) - \frac{1-\delta_i}{1-\delta_i(1-p_i)} \left(V(r_i) - \underline{u}_i \right) \quad \text{for every } r_i' \in \widehat{R}_i(r_i, s_i), \qquad \text{(D.2)}
$$

where the RHS is a lower bound on player i's payoff if he uses the strategy I described. When player i plays $a_i' \in A_i$ at (r_i, r_{-i}), the difference between his expected continuation value in the next period and $V(r_i)$ is

$$
\begin{aligned}
& p_e \sum_{s_i \in S_i} f_i(s_i | a_i', \sigma_{-i}^a(r_i, r_{-i})) \max_{r_i' \in R_i(r_i, s_i)} V(r_i') + p_n \sum_{s_i \in S_i} f_i(s_i | a_i', \sigma_{-i}^a(r_i, r_{-i})) V(r_i, s_i) \\
& + p_b \max_{r_i' \in R_i} V(r_i') + p_a \sum_{s_i \in S_i} f_i(s_i | a_i', \sigma_{-i}^a(r_i, r_{-i})) \max_{r_i' \in \widehat{R}_i(r_i, s_i)} V(r_i') - V(r_i). \qquad \text{(D.3)}
\end{aligned}
$$

If any in equilibrium, for any (r_i, r_{-i}) at which player i plays any $a_i' \neq a_i^*$ with positive probability in equilibrium, then by (D.1) and (D.2), the value of expression (D.3) is at least

$$
\frac{1-\delta_i}{\delta_i} c^* - (p_a + p_n) \frac{1-\delta_i}{1-\delta_i(p_a + p_n)} (\overline{u}_i - \underline{u}_i). \qquad \text{(D.4)}
$$

At any other (r_i, r_{-i}), inequality (D.2) implies that the value of expression (D.3) is at least

$$
-(p_a + p_n) \frac{1-\delta_i}{1-\delta_i(p_a + p_n)} (\overline{u}_i - \underline{u}_i). \qquad \text{(D.5)}
$$

Recall the definitions of V_k and π_k in the proof of Theorem 1. From (D.4) and (D.5), we know that

$$
V_{k+1} - V_k \geq \pi_k \frac{1-\delta_i}{\delta_i} c^* - (p_a + p_n)(\overline{u}_i - \underline{u}_i) \frac{1-\delta_i}{1-\delta_i(p_a + p_n)}. \qquad \text{(D.6)}
$$

For every $T \in \mathbb{N}$, summing up inequality (D.6) from $k = 1$ to $k = T$, we can obtain that

$$\sum_{k=1}^{T} \pi_k \leq \sum_{k=1}^{T} \left\{ (p_a + p_n) \frac{1-\delta_i}{1-\delta_i(p_a+p_n)} \cdot \frac{\overline{u}_i - \underline{u}_i}{c^*} + \frac{V_{k+1} - V_k}{c^*} \right\} \frac{\delta_i}{1-\delta_i} \leq T G_1 + G_0, \tag{D.7}$$

where $G_1 \equiv \frac{(p_a+p_n)(\overline{u}_i - \underline{u}_i)}{c^*} \cdot \frac{\delta_i}{1-\delta_i(p_a+p_n)}$ and $G_0 \equiv \frac{\overline{u}_i - \underline{u}_i}{c^*} \cdot \frac{\delta_i}{1-\delta_i}$. Since $\pi_k \in [0, 1]$ for every $k \in \mathbb{N}$, (D.7) implies that

$$\sum_{k=1}^{+\infty} (1-\overline{\delta}_i)\overline{\delta}_i^{k-1} \pi_k \leq (1-\overline{\delta}_i)^{\frac{G_0}{1-G_1}} + \overline{\delta}_i^{\frac{G_0}{1-G_1}} G_1 \leq (1-\overline{\delta}_i)^{\frac{G_0}{1-G_1}} + G_1. \tag{D.8}$$

By the law of total probabilities, the LHS is the average probability with which player i takes actions other than a_i^*. Fix any $\widehat{\delta}_i \in (0, 1)$ and let $\overline{\delta}_i \to 1$, the RHS of (D.8) equals G_1, and G_1 vanishes as $p_a + p_n \to 0$.

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