A New Approach to Robust Stability of Multiclass Queueing Networks
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**Robust Stability**
- Robust stability / Policy robustness
- Multiclass queueing networks
- Central design & decentralized control
  - Manage network parameters to ensure nothing bad happens
  - Each station is free to choose some policy parameters
- Focus on family of “fixed queue-ratio” policies
  - Aim: distribute a station’s workload among queues according to
    fixed ratios (collected in matrix $\Delta$)
  - Special case: static-priority policy; longest-queue policy

**Question:**
What are the sufficient conditions for robust stability?

**A Hierarchical Approach to Stability**

**Stability:** (\textsuperscript{[1]} Dai 1995)

The queueing network is stable if the fluid model is attracted to the origin in finite time.

**Observation:** (formalized in \textsuperscript{[2]} Delgado 2010)

This question of attraction to the origin can be decomposed into:
- The time the fluid model reaches state space collapse (SSC)
- The time until the Skorohod problem (SP) reaches the origin

- Full fluid model
  - 6-dimensional process (track workload of each class)
  - reach the proportions

- SSC
  - 3-dimensional process (track workload of each station)

- SP stability?

**Network Primitives**

**Reflection matrix $R$:** $R^{-1} = CMQ\Delta$

\begin{itemize}
  \item $M = \text{diag}(m_1, m_2, \ldots)$
  \item $m_k = \text{mean service time of class } k$
\end{itemize}

\begin{itemize}
  \item $P_{ki}$: probability that class $k$ job becomes a class $i$ job upon completion of service
  \item $\Delta_{kl}$: queue-ratio matrix
  \item $\delta_k$: if class $k$ is served in station $j$: $\Delta_{kj} = 0$, otherwise
\end{itemize}

By the nominal workload, ratios satisfy $\sum_{k \in E(i)} m_k \delta_k = 1$

**Stability via the Skorohod Problem** \textsuperscript{[3] Theorem 2.5 (Chen 1996)}

Assume that $R$ is completely-S, and let $\theta = R(p - \psi)$. Then the Skorohod problem is attracted to the origin if there exists a positive vector $\bar{h} \in R^j$ such that given any partition $(a, b)$ of $J$, $h_{-a}^{\top} \theta_b + R_{a,b} \Delta \bar{h} < 0$

for all $u \in \{v \in R^{|J|}; \theta_b + R_{a,v} \bar{v} = 0\}$. Call this property as Chen-S

**Robust Skorohod Problem Stability**

**Main Result:** “Convexity” in Policy

**Theorem 1** Suppose:

(i) For all static priority $\Delta$, $R_{a}^{-1} = CMQ\Delta$ is invertible with same-sign determinant;

(ii) For all static priority $\Delta$, $(R_{a}^{*})_{ab}$ is Chen-S.

Then, $\Delta_{a,b}^{*}$ is invertible and $(R_{a}^{*})_{ab}$ is Chen-S for any matrix $\Delta$.

**DHV network:** 8 static priority policies

**References**


**Robust Skorohod Problem Stability**

- Formulate an optimization problem to determine if Chen-S holds for fixed $\Delta$. Chen-S holds if optimal value $> 0$.
- Require significant analysis of the structure of $R$.

$P(\lambda) = \max_{\lambda > 0} \frac{\lambda}{\lambda - 1}$

**Uncertainty set:** family of values that $\Delta$ can obtain

**Theorem 2** “Line Convexity”

For any $\Delta_1$ and $\Delta_2$ that differ only in one station, if $(R_{a1}^{*}, \theta_{a1})$ and $(R_{a2}^{*}, \theta_{a2})$ are both Chen-S, then, for $\Delta = \lambda \Delta_1 + (1 - \lambda) \Delta_2$ with $\lambda \in (0,1)$, $(R_{a}^{*})_{ab}$ is also Chen-S.

**Example:** An Unbalanced DHV Network

$a = 0.8111$ (exogenous arrival rate)

$m_1 = 0.1, m_4 = 0.65$

$m_2 = 0.8, m_5 = 0.1$

$m_3 = 0.1, m_6 = 0.4$

* If $\Delta_1$ and $\Delta_2$ differ only in one station, then the segment between them is parallel to the axis.

* The region of $\Delta$ is a cuboid.

**Example:** Balanced DHV Networks

$m_1 + m_4 = 1, m_2 + m_5 = 1, m_3 + m_6 = 1$

**Robust State-Space Collapse (SSC)**

**Condition X**

\begin{itemize}
  \item All internal points (any fixed queue-ratio policies) have SSC
  \item Corner points (static priority policies) have SSC
\end{itemize}

For balanced DHV network, we can get SSC for any $\Delta$ if $m_1 + m_4 + m_6 < 2$