Motivation

- Evolution in time of a system of $K$ interacting particles seen as a network with $K$ nodes that send each other information at random times.
- Edges of the network represent the possibility of interaction, weights on the edges represent the amount of information sent when interacting.
- Very general framework that usually gives rise to models that are not computationally tractable.

Can we approximate the arrival processes to the different nodes as independent Poisson processes to obtain tractability, while preserving the geometry of the initial model?

Framework

- System of $K$ SDEs describing the evolution of the system. The times of interaction are given by point processes on $\mathbb{R}$ associated with each node.
- State of a node $i$ characterized by the intensity $(\lambda_i(t))_{t \in \mathbb{R}}$ of the associated point process $N_i (t)$ that is, by the $\mathcal{F}_t$-predictable process satisfying for all $s < t \in \mathbb{R}$:
  \[
  \mathbb{E}[N(s,t) \mid \mathcal{F}_s] = \int_s^t \lambda_i(s) \, ds \mathcal{F}_s,
  \]
  where $\mathcal{F}_t$ is the network history.
- For $1 \leq i \leq K$, state of node $i$ at time $t = \text{state of node } i \text{ at time } 0 + \text{arrivals from nodes } j \neq i \text{ up to time } t + \text{autonomous evolution of node } i \text{ up to time } t$.

A model from computational neuroscience

The Galves-Löcherbach (GL) model

\[
\lambda_i(t) = \lambda_0(0) + \sum_{j \neq i} \mu_{j,i} \int_0^t \lambda_j(s) \, ds + \int_0^t (\lambda_i(s) - \lambda_i(s^-)) N_i(s) \, ds,
\]
where $\lambda_i(t)$ is the intensity of the point process $N_i(t)$ at time $t$.

The replica-mean-field approach

- Given a network of $K$ interacting nodes, we consider $M$ identical distributed replicas (copies) of the initial network.
- If a node $i$ interacts with node $j$ in replica $m$ of $\{1, \ldots, M\}$, instead a replica is uniformly and independently chosen in $\{1, \ldots, M\} \setminus \{m\}$ (routing operation) and node $i$ interacts with node $j$ in the replica thus chosen.

Classical approach: the mean-field regime

- Study the behavior of the system when a certain parameter (typically the number $K$ of particles) goes to infinity.
- Approximate the effects of the interactions on a node by an empirical mean of the interactions over all the system.
- At the limit, this empirical mean typically converges to an expectation (theoretical mean) which allows for computational tractability.
- The geometry of the initial model is not preserved. In particular, the correlations between nodes due to the finite size of the system are lost.

Another tractable regime: the Poisson Hypothesis

Particles can be considered independent with arrivals to a node given by a Poisson point process with an intensity that preserves the mean number of interactions of the original network. Note that in the stationary framework, setting the Poisson Hypothesis is equivalent to ignoring Palm bias.

The replica-mean-field approach

- Intuitively:
  - The probability that two nodes in two distinct replicas interact is of order $\frac{1}{M} \to$ asymptotic independence
  - The arrivals process can be seen as a sum of $M$ rare events of order $\frac{1}{M} \to$ Poisson limit theorem.

Theorem

Main Tool: The Chen-Stein method

Let $N \in \mathcal{N}$ valued r.v. with finite positive mean $\rho$. Then $N$ has a Poisson distribution iff

\[
\rho \mathbb{E}[f(N + 1)] = \mathbb{E}[f(N)],
\]
for some rich enough class of functions. If this holds approximately for some r.v. $N$, then $N$ is approximately Poisson.

Proof sketch

- Prove some bounds for the exponential moments of $\Lambda_m$ and $\tilde{\Lambda}$, and some properties of the replica structure.
- Prove the following bound:
  \[
  \mathbb{E}[\Lambda_m] = \mathbb{E}[\tilde{\Lambda}],
  \]
  where $\mathbb{E}[\Lambda_m]$ is the expected value of $\Lambda_m$ and $\mathbb{E}[\tilde{\Lambda}]$ is the expected value of $\tilde{\Lambda}$.
- Use the mapping theorem. Then, for $\tilde{\Lambda}$, we have
  \[
  \mathbb{E}[\tilde{\Lambda}] = \mathbb{E}[\Lambda_m],
  \]
  and for $\Lambda_m$, we have
  \[
  \mathbb{E}[\Lambda_m] = \mathbb{E}[\tilde{\Lambda}],
  \]
  where $\Lambda_m$ is a random variable.

A triangular law of large numbers

The third point in the proof requires showing the following "triangular" law of large numbers (TLLN):

\[
\mathbb{E}\left[\sum_{m=1}^M \mathbb{E}[N_m] - N_m\right] \to 0
\]
when $M \to \infty$. However, $(N_m)$ are not i.i.d. random variables. Since convergence of arrivals depends on convergence of numbers of departures, consider the dynamics as the fixed point of some function $\Phi$ that associates to the law of the "input" the law of the solution to the RMP GL equations. If

- there is some kind of convergence to the fixed point of $\Phi$,
- the TLLN convergence is preserved by $\Phi$,
then, the TLLN convergence will also hold at the fixed point.

Some References