Continuous and discrete self-similarity

Let $d_f(x) = f(cx)$ denote the classical dilation operator on $\mathbb{R}^d$. Then, $(d_f)_{c>0}$ is a multiplicative semigroup, that is $d_{cf} = d_c d_f$ for all $c, f > 0$.

**Definition 1.** A Markov process $X$ on $\mathbb{R}^d$ is called $\phi$-self-similar ($\phi$-ss) if its semigroup $(P_t)_{t \geq 0}$ satisfies

$$P_d = d_P, \quad \forall c > 0, t \geq 0.$$ 

This reads: for each $c, t, x \geq 0$, as

$$cX(t, x) \overset{d}{=} X(ct, cx).$$

Can we define a dilation operator on $\mathbb{Z}$, with the same property as above? Note that $d_1(\mathbb{Z}) \subseteq \mathbb{Z}$. 

For $f: \mathbb{Z} \to \mathbb{R}$, let us define

$$D_f(n) = \frac{1}{n} \sum_{k=0}^{n-1} f(1-c^{-1}k) \in \mathbb{Z}. \tag{1}$$

When $c \in [0,1]$, $D_f(n) = f[B(n, c)]$. It can be easily verified that

$$d_A = A D_f \quad \forall c > 0$$

where $M(x) = E_f[(P_t x)(x)]$ with $P_t x$ a Poisson rv. (2) reveals the multiplicative structure of $D$: i.e. $D_{cd} = CD_{d_c}$ for all $c, d > 0$.

**Definition 2.** A Markov process $X$ on $\mathbb{Z}$ is said to be $D$-self-similar ($D$-ss) if its semigroup $(P_t)_{t \geq 0}$ satisfies

$$P_d = D_P, \quad \forall c > 0, t \geq 0.$$ 

When $c \in [0,1]$, the above definitions reads, for each $t \geq 0$ and $n \in \mathbb{Z}$, as

$$\text{Bin}(X(t, n), c) \overset{d}{=} X(t, \text{Bin}(n, c)).$$

Connections between $d$-ss and $D$-ss processes

From Lamperti [1], one gets that the set of $d$-ss processes with negative jumps is in bijection with the set $B$. In particular, the Dynkin characteristic operator is given by

$$G_d(f)(x) = \sigma^2 f''(x) + \int \{f(x) - f(x + 2t c f'(x))\} dP(x)$$

where $\sigma^2$ is defined in (3). Note that $G_d$, defined in (4) is a discretization of $G_d$.

**Theorem 2.** Let $X$ (resp. $X$) denote the $d$-ss (resp. $d$-ss) Markov process associated to $\phi \in B$.

1. (Scaling limit) $b_{\phi/n} \to \lambda \geq 0$ and $c_{\phi/n} \to \lambda > 0$ iff

$$\frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{\{X_n(k) > 0\}} \overset{d}{=} \mathbb{1}_{\{X(\lambda t, x) > 0\}}$$

2. (Gateways) Writing

$$P_t^\phi f(x) = E_f[X(t, x)],$$

we have, for all $t \geq 0$,

$$P_t^\phi = \mathbb{P}_t^\phi.$$ 

In other words, for any $t \geq 0$ and $x$, $P_t(X(t, x)) \to X(t, \text{Bin}(x, c))$.

Generalized $D$-Laguerre semigroups

For any $\phi \in B$, Patie and Savov [4] showed that $(K^\phi_\beta = P_t^\phi, t \geq 0)$ is an ergodic Feller semigroup on $C_0(\mathbb{R})$ with invariant distribution $\pi_0$. 

**Theorem 3.** For any $\phi \in B$, the following holds.

1. For all $\phi \in B$ and $t \geq 0$, $K^\phi_t = \mathbb{K}^\phi_t$ where $\mathbb{K}^\phi_t = P_t^\phi \mathbb{P}^\phi_t$.

2. $\mathbb{K}$ is an ergodic Feller semigroup whose invariant distribution is $\mathbb{P}^\phi_0(\Gamma(n + 1))$, $n \in \mathbb{R}$.

$\mathbb{K}$ is self-adjoint in $L^2(\mathbb{P}_0(\Gamma(n + 1))$, $n \in \mathbb{R}$.

Spectral expansion and hypocoercivity by gateway

From the gateway relation in Theorem 3.1 and [4], one gets the following.

**Theorem 4.** Let $\phi \in B$ be as in (3). If $\sigma^2 > 0$, the following holds.

1. (Spectral expansion) For all $t > 0$ and $f \in L^2(\mathbb{P}_0(\Gamma(n + 1))$, $n \in \mathbb{R}$,

$$\mathbb{K}^\phi_t f = \sum_{k=0}^\infty e^{-t k} \mathbb{V}^\phi_k n_k \mathbb{P}^\phi_t n_k,$$

where $\mathbb{K}^\phi_t n_k = e^{-t k} \mathbb{V}^\phi_k n_k$, $\mathbb{V}^\phi_k n_k = 1$(k=0).

2. (Hypocoercivity) There exists a constant $C_0 > 1$ such that

$$\|\mathbb{K}^\phi_t f - \mathbb{P}^\phi_0 f\|_{\mathbb{P}_0(\Gamma(n + 1))} \leq C_0 e^{-t \|f\|_{\mathbb{P}_0(\Gamma(n + 1))}} \forall t \geq 0$$

$C_0$ can be computed explicitly in terms of $\phi$.

Which Markov processes are $D$-self-similar?

Let $B$ be the class of all Bernstein functions $\phi$ which are of the form

$$\phi(u) = m + \sigma^2 u + \int 1 - e^{-x u} \mathbb{L}(x) dx,$$

where $m, \sigma^2 > 0$ and $\int \min\{x, r\} \mathbb{L}(x) dx < \infty$. For each $x \in B$, let us define the following operator on $\mathbb{R}^+$

$$G_\phi(f)(n) = \phi'(1) f(0) + \phi' f(n) + (m + \sigma^2) \phi(1) f(n) + \int \phi(1 - f(n + 1) + f(n + 1) \phi(1)) \mathbb{L}(x) dx.$$ 

This reads: for each $\phi \in B$, $(G_\phi, C_1(\mathbb{Z}))$ generates a discrete self-similar Feller semigroup on $\mathbb{Z}$.

**Theorem 5.** For any $\phi \in B$, $(G_\phi, C_1(\mathbb{Z}))$ generates a discrete self-similar Feller semigroup on $\mathbb{Z}$.

**Conjecture:** The generator of any $D$-ss Feller process is of the form $G_\phi$ for some $\phi \in B$.

Interweaving with diffuse Laguerre semigroups

**Definition 3.** (Miloś & Patie [2], 2022) Two Markov processes $P, Q$ are called interweaved if there exist two Markov kernels $\Lambda, \Lambda'$ such that for all $t \geq 0$,

$$P_t A = Q_t \Lambda A, \quad Q_t A' = P_t \Lambda' A$$

for some non-negative random variable $\tau$. We use the notation $P \overset{\tau}{=} Q$.

In [2], the authors showed that the entropy decay, hypercontractivity, ultracontractivity, and cut-off phenomenon are preserved under interweaving relationship.

**Theorem 5.** Let $\phi \in B$ be such that $\sigma^2 > 0$ and $\int e^{r \mathbb{L}(r)} < \infty$. Then, there exists an explicit infinite divisible rv $\tau$ such that $K^\phi \overset{\tau}{=} K$.

Entropy and hypercontractivity by interweaving

The self-adjoint semigroup $(K^\phi_t)_{t \geq 0}$ satisfies the log-Sobolev inequality. Being self-adjoint, it entails the entropy convergence and the hypercontractivity. The semigroups $K^\phi$ does not satisfy the log-Sobolev inequality, and are not self-adjoint. However, the interweaving relation in Theorem 5 yields the following.

**Theorem 6.** Let $\phi \in B$ be as in Theorem 5. Then the following holds.

1. (Entropy decay) For any initial distribution $\mu$ on $\mathbb{Z}$, and $t \geq 0$, one has

$$\text{Ent}(\mathbb{K}^\phi_t \mu) \leq C e^{-t \text{Ent}(\mu)}$$

2. (Hypercontractivity) For all $t \geq 0$ and $f \in L^2(\mu)$,

$$\|\mathbb{K}^\phi_t f\|_{L^\infty(\mu)} \leq C e^{-t \|f\|_{L^2(\mu)}}, \quad p(t) = 1 + e^t$$

References