The Hare and the Tortoise Problem- [Sheng 1978]

Solve:
\[
\max \mathbb{E} \left[ \int_0^\infty e^{-t} X_t \, dt \right],
\]
\[
dX_t = b_t \, dt + \sigma_t \, dZ_t + L_t,
\]
where \(Z_t\) is a standard Brownian motion, \(L_t\) is the Skorokhod term and 
\[
(b_t, \sigma_t) = \begin{cases} (b_1, \sigma_1), \\ (b_2, \sigma_2). \end{cases}
\]
Solved by Sheng: if \(b_1 < b_2\) and \(\sigma_1 > \sigma_2\) the choice of \((b_t, \sigma_t)\) depends on the condition \(X_t \leq x\) for some \(x \in \mathbb{R}\).

Main Results
- We fully solve the asymptotic game,
- the optimal SC control: \(c^\mu\),
- the choice of the adversary is based on the first two moments only

\[
\text{the asymptotic game’s value is the unique solution to the HJB equation}
\]
\[
\sum_{\ell=1}^L \max_{(b, q)} \{ u'(w) b + u''(w) q \} - u(w) + w = 0, \quad w \geq 0.
\]

Finite Uncertainty Class
- A finite uncertainty class translates into a finite number of points in \(\mathbb{R}^2\),
- these can be reduced to a dominating subset (red squares below).

Proof Steps
- The workload and the queue length are proportional (Reiman’s snapshot principle) and both concentrate asymptotically on one buffer,
- the limits of the processes of our system form an admissible solution to a suitable martingale control problem,
- asymptotically optimal strategy for the adversary is identified.