Choice over Assessments

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October 2023
Section 1

Introduction
Selecting assessments

• **Assessment** is lottery over *scores* which depends on agent’s type
• Scores reveal information about agent’s type
• Agent choose assessment to increase expected score (e.g., SAT vs ACT)

This is not choice *under* uncertainty. *It is choice of uncertainty.*
Assortative matching intuition

Intuitively, higher types prefer more accurate assessments:

- Lowest type wants assessment that reveals no information
- Highest type prefers perfectly revealing assessment

Want to formalize and study this intuition for comparing assessments.
Roadmap

• Model
• Assortative matching result
• Relationship to other orders
• Menu design and applications
• Extensions and repeated testing
Section 2

Model
Model

- Agents have private types $\theta \in \Theta$ distributed by $G$
- Scores, $s \in S$, distributed by assessments, $F_i$, conditional on type
- Agent’s utility over scores, $u$, weakly increasing
- Agent payoff is $U(i, \theta) = \int_S u(s) dF_i(s|\theta)$ from choosing assessment $F_i$
- $I_\theta := \arg \max_i U_{i \in I}(s, \theta)$ denotes the set of assessments that type $\theta$ prefers
**Definition of types/assessments**

Higher types FOSD lower types’ distributions for each assessment

**Assumption (type order)**

For all assessments, \( F_i, s \in S \) and all \( \theta, \theta' \in \Theta \) with \( \theta < \theta' \),

\[
F_i(s|\theta') \leq F_i(s|\theta)
\]
Decreasing differences property

**Definition (decreasing differences)**
Assessments satisfy DD (submodularity) iff for all $s \in S$, $i, j \in I$ with $i < j$ and $\theta, \theta' \in \Theta$ with $\theta < \theta'$,

$$F_j(s|\theta') - F_i(s|\theta') \leq F_j(s|\theta) - F_i(s|\theta)$$

We will see DD is sufficient for weak assortative matching.
Section 3

Assortative matching result
Theorem

DD holds if and only if the expected utility

$$U(i, \theta) = \int_{s \in S} u(s) dF_i(s|\theta)$$

is supermodular for any monotone utility function.

Corollary

DD implies $I_{\theta'}$ strong-set order dominates $I_{\theta}$ for all $\theta' > \theta$. 
Example: Normal Distributions

\[ F_2(s|\theta_L) = F_1(s|\theta_H) \]

Scores \( s \)

\[ F_2(s|\theta_L) \]

\[ F_1(s|\theta_L) = F_1(s|\theta_H) \]

\[ F_2(s|\theta_H) \]
Example: Normal Distributions

\[ F_2(s|\theta_L) - F_1(s|\theta_L) \]
\[ F_2(s|\theta_H) - F_1(s|\theta_H) \]
Suppose $F_2$ reveals the agent’s type with certainty while $F_1$ is uniform independently of type. For any $\theta < \theta'$,

$$F_2(s|\theta') - F_1(s|\theta') = 1_{s \geq \theta'} - s \leq 1_{s \geq \theta} - s = F_2(s|\theta) - F_1(s|\theta)$$
Example II

Assume a family \( \{ F_\alpha(\cdot|\theta) : \alpha \in [0, 1] \} \) of cdfs of distributions that, with probability \( \alpha \), perfectly reveals the agent’s type and, with probability \( 1 - \alpha \), draws a random score from the \( \mathcal{U}[0, 1] \) distribution. Then,

\[
F_\alpha(s|\theta) = 1_{\{s \geq \theta\}} \alpha + s(1 - \alpha)
\]

Now fix \( \alpha' > \alpha \) and \( \theta' > \theta \). Then,

\[
F_{\alpha'}(s|\theta') - F_\alpha(s|\theta') = \left(1_{\{s \geq \theta'\}} - s\right)(\alpha' - \alpha)
\leq \left(1_{\{s \geq \theta\}} - s\right)(\alpha' - \alpha)
= F_{\alpha'}(s|\theta) - F_\alpha(s|\theta).
\]

In this case, a higher assessment corresponds to a higher \( \alpha \). Here, our ordering coincides with Blackwell informativeness. We will see later that this is not always the case.
Section 4

Relationship to other orders
Lemma
If $S := \{s_L, s_H\}$, the Blackwell informativeness criterion implies DD.

Proof.
Suppose assessment $i$ is a garbling of assessment $j$:

$$F_j(s_L|\theta') - F_i(s_L|\theta') = p_j(s_L|\theta')(1 - z(s_L, s_L)) - z(s_L, s_H)p_j(s_H|\theta')$$

$$\leq p_j(s_L|\theta)(1 - z(s_L, s_L)) - z(s_L, s_H)p_j(s_H|\theta) = F_j(s_L|\theta) - F_i(s_L|\theta).$$
Relationship with Blackwell (2 scores)

Blackwell is sufficient for DD, but not necessary. Consider $P_i$ and $P_j$ s.t.

\[
\begin{align*}
    p_i(s_L|\theta) &= 1 - \epsilon & p_i(s_L|\theta') &= \frac{1}{2} \\
    p_j(s_L|\theta) &= \frac{1}{2} & p_j(s_L|\theta') &= 0
\end{align*}
\]

assessment $i$ is not a garbling of $j$ for $\epsilon < \frac{1}{4}$. Yet, DD is satisfied:

\[
\underbrace{F_j(s_L|\theta) - F_j(s_L|\theta')}_{\frac{1}{2}} \geq \underbrace{F_i(s_L|\theta) - F_i(s_L|\theta')}_{\frac{1}{2} - \epsilon}
\]
Blackwell does not imply DD with 3 or more scores

In general, Blackwell does not imply DD

Intuitively, a medium type may care more about accuracy than a high type if the difference in utility from a medium and low score is sufficiently large. 

Counterexample
Definition (Concordance ordering)
assessment $j$ dominates $i$ in the concordance ordering iff $F_j(s) = F_i(s)$ and

$$p_j(S \leq s, \Theta \leq \theta) \geq p_i(S \leq s, \Theta \leq \theta)$$

If the marginals are the same ($F_j(s) = F_i(s)$) DD implies the concordance ordering. The converse is true if there are only two scores. ✨Proof
Relationship with concordance ordering

Because the underlying distribution of types does not depend on the assessment chosen, we can divide both sides to get a definition in terms of conditionals:

\[ F_j(s|\Theta \leq \theta) \geq F_i(s|\Theta \leq \theta) \]

Because our problem is two dimensional, the concordance ordering is equivalent to greater weak association, the supermodular ordering, the convex-modular ordering, and the dispersion ordering.
Section 5

Menu design and applications
Collecting information

If we do not use the information, we can collect types:

- Construct a menu of garblings in the DD order
- Obtain types from observing the choice of assessment

However, this does not allow use of types in a way that affects agents.
Menu design motivation

Can we design assessment menus to make scores more accurate?

Sort of.

- Use assortative matching to reveal information
- Need additional assumptions to misalign preferences of principal/agent
Simplest example

Professor is writing graduate admissions letters for undergrads

- Has assessment with three scores: 1, 2, 3
- Students have two types: $\theta_L, \theta_H$
- Assume student utility, $u$, is concave
- Professor wants to write letters for $\theta_H$ only
- Assessment usually assigns $\theta_L$ to 1, but sometimes assigns 2 or 3

With this assessment, professor must occasionally be writing letters for $\theta_L$. 
Simplest example

Professor offers a menu of assessment and garbling that only gives score 2

- Students with $\theta_L$ will take the garbling
- Any student with score 3 must have type $\theta_H$
- Professor can write letters for $\theta_H$ only

Note: We used concavity of $u$ to ensure that students do not also only care about score 3. If they did, any menu would be detrimental.
Section 6

Extensions and repeated testing
Choice of assessments under repetition

Suppose the agent may retake assessments at cost $c$

- **New question:** How does her choice of assessment change?
- This is now an *optimal stopping/search problem*.

Consider type $\theta$. Suppose she chooses assessment $i$ because she finds it preferable to any other assessment. Assume she has a current best score of $s^*$ and is considering whether to stop.

Assume each trial costs $c$, and that $U(i, \theta) - c > u(s)$ for all $i \in I$ and all $\theta \in \Theta$. 

If continuing is preferable, then the value of doing so is

$$V_i(s^*, \theta) = (1 - F_i(s^*|\theta))E[\max\{u(s), V_i(s, \theta)\} | s > s^*] + F_i(s^*|\theta)V_i(s^*, \theta) - c$$

$$\Rightarrow V_i(s^*, \theta) = E[\max\{u(s), V_i(s, \theta)\} | s > s^*] - \frac{c}{(1 - F_i(s^*|\theta))}$$

The value of stopping is simply $u(s^*)$. Thus, type $\theta$ stops at $s^*$ if and only if

$$E[u(s) | s > s^*, \theta, i] - \frac{c}{(1 - F_i(s^*|\theta))} \leq u(s^*)$$

$$\Rightarrow \int_{s > s^*} u(s)dF_i(s|\theta) - c \leq u(s^*)$$

We let $s^*_\theta := \arg\max_{s^*} \left\{ \int_{s > s^*} \frac{u(s)dF_i(s|\theta) - c}{(1 - F_i(s^*|\theta))} \leq u(s^*) \right\}$ denote the set of optimal stopping scores for type $\theta$ at assessment $i$. Note that $\theta' > \theta \iff s^*_{\theta'_i} \geq s_{\theta_i}$. 
Let:

\[ U^*(i, \theta) := \int_{s \in S} u(s) dF_i(s|\theta, s > s^*_i) - \frac{c}{(1 - F_i(s^*|\theta))} \]

It is necessary and sufficient for the supermodularity of \( U^* \) that, for \( j > i \) and \( s \geq \max_{\tilde{\theta}, k} \{ s^*_j \} \),

\[ F_j(s|\theta', s > s^*_{j|\theta}) - F_i(s|\theta', s > s^*_{i|\theta}) \leq F_j(s|\theta, s > s^*_j) - F_i(s|\theta, s > s^*_i) \]

since the total expected costs are decreasing in type.
Example: repeated assessments with low costs

Suppose that $c$ is low enough that all players choose a $ar{s}$ as their cutoff. Then, weak assortative matching is equivalent to

$$\frac{p_i(\bar{s}|\theta_L) - p_j(\bar{s}|\theta_L)}{p_i(\bar{s}|\theta_L)p_j(\bar{s}|\theta_L)} \geq \frac{p_i(\bar{s}|\theta_M) - p_j(\bar{s}|\theta_M)}{p_i(\bar{s}|\theta_M)p_j(\bar{s}|\theta_M)} \geq \frac{p_i(\bar{s}|\theta_H) - p_j(\bar{s}|\theta_H)}{p_i(\bar{s}|\theta_H)p_j(\bar{s}|\theta_H)}$$

Because of the type definition, this is implied by

$$p_i(\bar{s}|\theta_L) - p_j(\bar{s}|\theta_L) \geq p_i(\bar{s}|\theta_M) - p_j(\bar{s}|\theta_M) \geq p_i(\bar{s}|\theta_H) - p_j(\bar{s}|\theta_H)$$

which is implied by DD.
Thank You!
Section 7

Proofs
Sufficiency of DD

Proof.
Assume $j \in \mathcal{I}_\theta$ and let $i < j$. If $i \in \mathcal{I}_\theta'$, then, using integration by parts,

$$0 \leq \int_{s \in S} u(s) dF_i(s|\theta') - \int_{s \in S} u(s) dF_j(s|\theta')$$

$$= \left( u(\bar{s}) - \int_{s \in S} F_i(s|\theta') du(s) \right) - \left( u(\bar{s}) - \int_{s \in S} F_j(s|\theta') du(s) \right)$$

$$= \int_{s \in S} (F_j(s|\theta') - F_i(s|\theta')) du(s)$$

$$\leq \int_{s \in S} (F_j(s|\theta) - F_i(s|\theta)) du(s)$$

$$= \int_{s \in S} u(s) dF_i(s|\theta) - \int_{s \in S} u(s) dF_j(s|\theta)$$

Since $\theta$ prefers $j$, the above implies that $\theta$ must also prefer $i$, i.e, $i \in \mathcal{I}_\theta$. ■
Necessity of DD

Proof.
Suppose, by means of contradiction, that DD is violated. That is, there exists $s^*$ such that

$$F_j(s^*|\theta') - F_i(s^*|\theta') > F_j(s^*|\theta) - F_i(s^*|\theta)$$

(1)

Consider the following weakly monotone utility function:

$$u(s) = \begin{cases} 
0 & \text{if } s < s^* \\
1 & \text{if } s \geq s^*
\end{cases}$$

Then the expected utility from assessment $k$ for type $\theta$ is $1 - F_k(s^*|\theta)$. By (1) SM of the expected utility is violated because:

$$EU_j(\theta') - EU_i(\theta') < EU_j(\theta) - EU_i(\theta)$$
Blackwell counterexample

With three scores, Blackwell does not imply DD. To see why, consider $S := \{s_L, s_M, s_H\}$, $\Theta = \{\theta_M, \theta_H\}$ and $u(s_L) < u(s_M) = u(s_H)$. Let assessment $j$ be perfectly revealing, i.e., $p_j(s_M|\theta_M) = p_j(s_H|\theta_H) = 1$ and let assessment $i$ be a garbling of $j$ where

$$p_i(s_L|\theta_M) = p_i(s_M|\theta_L) = p_i(s_M|\theta_H) = p_i(s_H|\theta_H) = \frac{1}{2}$$

Then, type $\theta_M$ really wants to avoid getting $s_L$, whereas type $\theta_H$ doesn’t have to worry about it since it has no chance of obtaining it. Note that the example above violates the condition in DD:

$$F_j(s_L|\theta_M) - F_i(s_L|\theta_M) = -\frac{1}{2} < 0 = F_j(s_L|\theta_H) - F_i(s_L|\theta_H)$$
Sufficiency of concordance ordering

Proof.

\[ E_\theta \left[ F_j(s|\tilde{\theta}) - F_i(s|\tilde{\theta}) | \tilde{\theta} \leq \theta \right] \Pr(\tilde{\theta} \leq \theta) \]

\[ + E_\theta \left[ F_j(s|\tilde{\theta}) - F_i(s|\tilde{\theta}) | \tilde{\theta} > \theta \right] \Pr(\tilde{\theta} > \theta) = 0 \] (2)

\[ \Rightarrow E_\theta \left[ F_j(s|\tilde{\theta}) - F_i(s|\tilde{\theta}) | \tilde{\theta} > \theta \right] \leq 0 \] (3)

\[ \Rightarrow \int_{\theta \in \Theta} \left( F_j(s|\tilde{\theta}) - F_i(s|\tilde{\theta}) \right) dF(\tilde{\theta}|\tilde{\theta} > \theta) \leq 0 \]

\[ \Rightarrow F_j(s|\tilde{\theta} > \theta) - F_i(s|\tilde{\theta} > \theta) \leq 0 \]

Where we used \( F_i(s) = F_j(s) \) in line (2) and Definition 1 to derive (3). ■